5-2013

Soliton Solutions of the Nonlinear Schrödinger Equation.

Erin Middlemas
East Tennessee State University

Department of Physics and Astronomy Accepted by the faculty of the

Department of Physics and Astronomy, East Tennessee State University, in partial fulfillment of the requirements for the Degree of Bachelor of Science.

FACULTY COMMITTEE:

Soliton Solutions of the Nonlinear Schrödinger Equation

A Project

Presented to

the faculty of the Undergraduate College of

Arts and Sciences

East Tennessee State University

In Partial Fulfillment

of the Requirements for the Degree

Bachelor of Science

by

Erin Middlemas

April 7, 2013

Acknowledgments

I would like to thank my thesis advisor Dr. Jeff Knisley for his patience and guidance throughout this project. His enthusiasm and appreciation for science not only gave me the motivation to pursue a thesis in nonlinear dynamical systems but also enlightened me of intriguing subjects such as Bose-Einstein Condensates, particle physics, and string theory. I am extremely grateful for all of his advice and thorough help throughout the preparation of my thesis.

Abstract

The nonlinear Schrödinger equation is a classical field equation that describes weakly nonlinear wave-packets in one-dimensional physical systems. It is in a class of nonlinear partial differential equations that pertain to several physical and biological systems. In this project we apply a pseudo-spectral solution-estimation method to a modified version of the nonlinear Schrödinger equation as a means of searching for solutions that are solitons, where a soliton is a self-reinforcing solitary wave that maintains its shape over time. The pseudo-spectral method estimates solutions by utilizing the Fourier transform to evaluate the spatial derivative within the nonlinear Schrödinger equation. An ode solver is then applied to the resulting ordinary differential equation. We use this method to determine whether cardiac action potential states, which are perturbed solutions to the Fitzhugh-Nagumo nonlinear partial differential equation, create soliton-like solutions. After finding soliton-like solutions, we then use symmetry group properties of the nonlinear Schrdinger equation to explore these solutions. We also use a Lie algebra related to the symmetries to look for more solutions to our modified nonlinear Schrödinger equation.

1. Introduction

While linear partial differential equations (PDEs) give rise to low-amplitude waves that occur frequently in the physical world [1], nonlinear waves with non-dispersive traits and soliton-like properties can occur naturally also. Soliton-like properties have been observed in water waves, fiber optics, and biological systems such as proteins and DNA [2, 1, 3, 4]. Since linear PDEs fail to take into account phenomena produced by non-linearity, other mathematical models are needed. Nonlinear PDEs such as the Kortweig de Vries equation and the nonlinear Schrödinger equation [5] are used to describe the characteristics of these waves more accurately.

Cardiac action potentials (CAPs) also display soliton-like properties. Cardiac cells, like neuron and muscle cells, are excitable cells and are electrically charged by having the membrane act as a capacitor. Previous research [6] has shown CAPs to be well-fit by solutions to the Fitzhugh-Nagumo model,

$$u_t = u_{xx} + u(1-u)(a-u) - w \tag{1.1}$$

$$w_t = \varepsilon(u - \gamma w), \tag{1.2}$$

where $u(x,t)$ and $w(x,t)$ are the fast and slow voltage responses at time t and distance x from origin of the CAP, γ is the rate of decay of the slow signal when ε is small, and a is the voltage threshold parameter. These two equations account for the discharging of the membrane and the recovery of this charge. If $a = 1$ and $\varepsilon = 0$, the fast voltage response is a traveling wave of the form

$$u(x,t) = f(x - 2kt),$$

where

$$f(x - 2kt) = \frac{1}{1 + Pe^{-(x-2kt)}} \tag{1.3}$$

where P is a constant term. Characteristics of these traveling waves lead us to believe soliton waves that are solutions to a perturbed nonlinear Schrödinger equation could also describe CAPs.

In this paper, we explore the soliton-like properties of solutions to the Fitzhugh-Nagumo model. We first introduce cardiac action potentials, nonlinear partial differential equations, and the reasoning behind the methods of our research. Next, we explain the procedure by which we find our perturbed solution describing CAPs and the perturbed nonlinear Schrödinger equation. Symmetries of our perturbed nonlinear Schrödinger equation are then discussed. We last provide results and discussion for both numerical and theoretical work, concluding with some future goals.

1.1 Cardiac Action Potentials

Cardiac Arrhythmia is a common problem in which there is abnormality in the normal sequence of the heart's electrical impulses. This leads to an abnormal heart beat [7]. Although arrhythmia can be nonthreatening, there are some cases in which the condition is more serious and can even be fatal. Due to these cases, it has been necessary to research and study cardiac arrhythmia in order to understand the condition more. One efficient tool for researching arrhythmia is through mathematical models describing cardiac action potentials. These mathematical models, such as the Fithugh Nagumo Model, take into account the creation of electrical impulses within the heart. Studying the exchange of ions may help to develop pharmaceutical drugs to treat cardiac arrhythmia [8].

A cardiac action potential is an electrical signal that allows an excitable cell to communicate with other cells [9, 10]. As previously stated, the potential is created by the membrane acting as a capacitor. The movement of ions inside and outside of the cell membrane is controlled, creating a potential difference across the membrane. As ions transfer into the cell, the membrane depolarizes, thus producing an action potential. A cardiac action potential will not occur until the membrane within a cell reaches a particular threshold value. This prevents random triggerings of electrical impulses. When a cardiac cell is excited and discharges, the charge is sent to a neighboring cell. This creates the charge difference needed for a cardiac action potential.

The cardiac action potential has 4 phases. Phase 0 consist of the cell reaching the threshold potential and then having Na^+ ions rush into the cell, causing a rapid depolarization. The reaction of the Na^+ channels closing and K^{++} channels activating is considered phase 1. This phase reduces membrane potential and re-polarizes the cell. Phase 2 begins as Ca^{2+} channels open while K^+ is still leaving the cell, causing a plateau phase in CAPs. Lastly, phase 3 consists of the completion of re-polarization as Ca^{2+} channels are closed and the potential within the membrane returns to its normal value.

There are two types of cardiac cells: pacemaker cardiac cells and nonpacemaker cardiac cells. While pacemaker cells are responsible for pacing the heart, the work within this paper is directed towards nonpacemaker cardiac cells. These cells consist of two types: Purkinje fiber cells and myocardiac cells. These cell types have similar

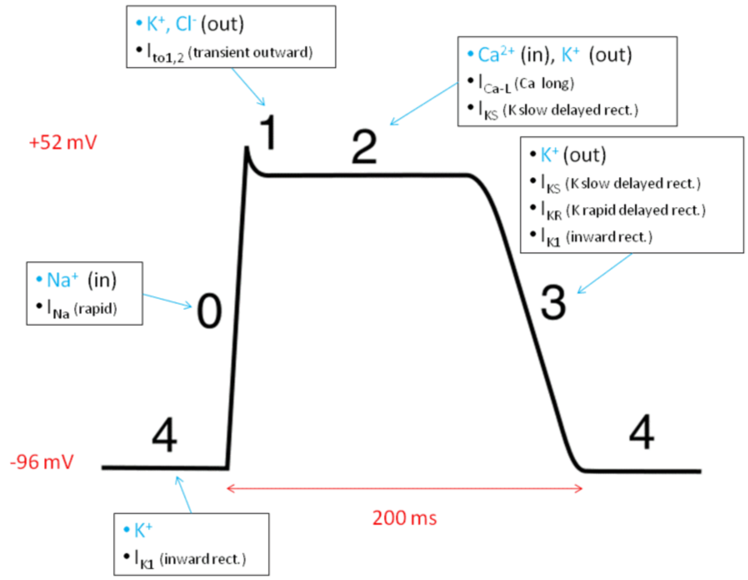


Figure 1.1: Phases within a ventricular myocyte action potential [11].

action potentials and are the potentials described by the mathematical models in the next section.

1.1.1 Mathematical Models of CAPs: The Fitzhugh Nagumo Equation

Alan Lloyd Hodgkin, Andrew Fielding Huxley, and Bernard Katz were the first people to describe action potentials by a mathematical model. The model, named the Hodgkin Huxley system, describes 4 processes [11]:

1. Change in the membrane potential
2. Potassium activation
3. Sodium activation
4. Sodium inactivation.

Soon after, the cardiac action potential was specifically studied, first by Denis Noble in 1962 as an extension of the Hodgkin Huxley system [12]. Soon many models of the cardiac action potential followed, each model accounting for new discoveries of the

CAP. Thus, science has given numerous mathematical models to describe our system. The Fitzhugh Nagumo Model is a particular partial differential equation (PDE) that is simpler than the highly nonlinear Hodgkin Huxley system. Richard Fitzhugh was a biophysicist in the 1960s who developed the model by constructing an electrical circuit which consisted of a capacitor, a tunnel diode, a resistor, inductor, and a battery in series. The Fitzhugh Nagumo PDE consists of 2 equations that can be derived from the Hodgkin Huxley system [8, 12].

1.2 Nonlinear Partial Differential Equations

Consider the swaying of a crowd during a football game. We can describe this motion as a wave moving through the crowd. Sometimes, however, the wave can build up along barriers within the crowd, and dissipate due to the instability. These barriers are what cause non-linearity within the wave [1].

The simplest partial differential equation (PDE) to describe the motion of a nonlinear wave is the following:

$$u_t + 2uu_x = 0 \tag{1.4}$$

This is similar to the equation for a linear wave except that now the velocity, $c = ku$, is dependent on the size of the function u. Nonlinear PDEs can differ significantly from linear PDE's. For example, the superposition principle does not hold for nonlinear PDEs. Solutions in the form of nonlinear waves are important in modeling the interaction of light with matter, the formation of galaxies, and chemical reactions [1].

1.2.1 Nonlinear PDEs and Solitons

Solitary waves are waves that are localized within a region and retain their form over a certain period of time [4]. These structures have the ability to pass through other waves with only a change of phase. Solitons are solitary waves that are also solutions to completely integrable partial differential equations. Solitons obtain their stability through a delicate balance between non-linearity and dispersion[1].

The discovery of solitary waves can be credited to John Scott Russell (1808-1882), who encountered a soliton wave traveling through the water on the Union Canal in Scotland. He followed the wave on horseback, observing the wave's maintenance of shape while traveling. Russell then began to study soliton waves by conducting experiments within a wave tank. He made the following observations [13]:

1. Solitons maintain their shape while traveling at a constant speed
2. They are localized within a region at any given time.
3. They can pass through other waves with no change in amplitude, velocity, or shape.

Figure 1.2: A barchan dune is an example of a soliton-like object. As these dunes drift, the smaller dunes can pass through the larger ones and appear on the other side. However, unlike a true soliton, the sand particles within the dunes do not pass through each other [18].

Perhaps the most widely known nonlinear PDE with soliton solutions is the Korteweg-de Vries (KdV) equation [1].

Figure 1.3: A solitary water wave [14].

The KdV equation derived its name from the Dutch mathematicians Diedrick Korteweg and Gustav de Vries. The history of the KdV equation roots to John Scott Russell and his observation of the first ever reported solitary wave of constant form at the Union Canal in Hermiston. This event led to the careful study of wave translation. In the 1870's the shape of the solitary wave was derived by Boussinesq and Lord Rayleigh as a traveling wave of the form [13]

$$r(x,t) = \alpha \operatorname{sech}^2[\beta(x-ct)], \tag{1.5}$$

from which the KdV equation was finally derived from in 1895. In this paper we will analyze soliton solutions of a variation of the nonlinear Schrödinger (NLS) equation. The nonlinear Schrödinger equation can be derived from the Korteweg-de Vries equation [1].

1.3 The Nonlinear Schrödinger Equation

1.3.1 Deriving the Nonlinear Schrödinger Equation

The NLS equation is in the form of:

$$iu_t = -u_{xx} + 2k|u|^2u, \tag{1.6}$$

where k is a constant term. The NLS accounts for the slow time and space modulation of the amplitude of a basic linear wave that is created through variations in the medium and nonlinear effects. The non-radiating solutions to the NLS are solitons.

To derive the NLS, we begin with the KdV equation:

$$U_t + U_{xxx} + f(U)U_x = 0, \tag{1.7}$$

We first expand the function $f(U)$ as a series in U:

$$f(U) = c_1U + c_2U^2 + c_3U^3 + ... \tag{1.8}$$

We want our equation to describe a low-amplitude, slowly varying wave packet. An approximate solution is given by

$$U \sim \left[u(x,t)e^{i(k_0x - w_0t)} + c.c.\right] \tag{1.9}$$

where w_0 is the dispersion relation of the Korteweg-de Vries equation(KdV),k_0 is the wavenumber, u is the function that is slowly varying in space and time, and c.c. are the complex conjugates of $ue^{i(k_0x-w_0t)}$. We can expect u to move at a group velocity of

$$V_0 = w'(k_0) = -3k_0^2, \tag{1.10}$$

where

$$w(k_0) = -k_0^3. \tag{1.11}$$

This group velocity depends on different time and space variables with the slow time and space variables being

$$x' = \varepsilon(x - v_0 t), \qquad t' = \varepsilon^2 t. \tag{1.12}$$

We then use a perturbation series with ε being the perturbation parameter. That is,

$$U = U_0 + \varepsilon U_1 + \varepsilon^2 U_2 + ..., \tag{1.13}$$

where $U_j(x,t)$ are the series coefficients of

$$U = u(x', t')e^{i\theta} + c.c. \tag{1.14}$$

We insert these expressions into the KdV equation and obtain

$$\begin{aligned}
\varepsilon^2(u_{1t} + u_{1xxx}) + \varepsilon^3(u_{2t} + u_{2xxx}) + O(\varepsilon^4) \quad = \quad & -\varepsilon^2 c_1 \left(ik_0 u^2 e^{2i\theta} + c.c.\right) - \\
& \varepsilon^3 \left[(u_{t'} + 3ik_0 u_{x'x'})e^{i\theta} + c.c.\right] - \varepsilon^3 c_1 \left[u u_{x'} e^{2i\theta}\right. \\
& \left. + c.c. + \left(|u|^2\right)\varepsilon\right] - \varepsilon^3 c_1 \left[(ue^{i\theta} + c.c.)U\right]_x \\
& -\varepsilon^3 c_2 \left(ue^{i\theta} + c.c.\right)^2 \left(ik_0 u e^{i\theta} + c.c\right) + O\left(\varepsilon^4\right).
\end{aligned}$$

We next manipulate the dispersion term and the group-velocity formula to cancel out certain terms. The equation then becomes

$$U_{1t} + U_{1xxx} = -ic_1 k_0 u^2 e^{2i\theta} + c.c. \tag{1.15}$$

where U_{1t} and U_{1xxx} are derivatives with respect to the fast variables x and t. The solution to U_1 is

$$U_1 = u_{12}(x', t')e^{2i\theta} + c.c. + u_{10}(x', t'). \tag{1.16}$$

where $u_{12} = \frac{ck_0}{2w_0+8k_0^3}u^2$. Substituting into the KdV equation yields

$$U_{2t}+U_{2xxx} = -e^{i\theta}\left[u_{t'} + 3ik_0 u_{x'x'} + \frac{i{c_1}^2{k_0}^2}{2w_0 + 8k_0^3}|u|^2 u\right] + ic_1 k_0 u_{10} u + ic_2 k_0 |u|^2 u + O(\varepsilon^2). \tag{1.17}$$

By supposing $u_{10} = \frac{c}{\nu_0}|u|^2$ and letting

$$iu_{t'} - 3k_0 u_{x'x'} + \alpha|u|^2 u = 0 \tag{1.18}$$

be the coefficient of $e^{i\theta}$ in (1.17), we finally obtain the nonlinear Schrödinger equation for a wave packet. Equation (1.18) is of the form

$$iu_t = -u_{xx} + \alpha|u|^2 u \tag{1.19}$$

with

$$\alpha = \frac{c_1^2}{6k_0} - c_2 k_0. \tag{1.20}$$

Equation (1.19) is called the nonlinear Schrödinger equation because of its close resemblence to a Schrödinger equation. Indeed, if $\alpha = 0$, the NLS is the Schrödinger equation of a free particle. However as this derivation shows, the NLS is better understood physically as an equation for slowly modulated wave packets such as solitons.

1.4 CAPs and the Gross-Pitaevskii Equation

Due to the interaction of the fast and slow excitation variables within the Fitzhugh Nagumo Model, there is reason to believe that cardiac action potentials are soliton-like. If we can show that perturbed solutions of the Fitzhugh-Nagumo model are solutions to a perturbed NLS, then we have evidence to support that CAPs are solitons. Perturbed solutions of the Fitzhugh-Nagumo model can then be used to find a family of solutions to a Gross-Pitaevskii equation,

$$iU_t = -U_{xx} + 2k|u|^2 u + \Phi(x,t,u)u \tag{1.21}$$

where $\Phi(x,t,u)$ is a potential function. After finding the perturbed solutions for a suitable choice of the potential, a pseudo-spectral method is used to numerically determine the properties of the resulting waves. The closed-form solutions to the

Gross-Pitaevskii equation are then utilized to generate more solutions.

It should be noted (1.21) with $\Phi(x,t) = 0$ reduces to (1.19). Also if $k = 0$, (1.21) reduces to the linear Schrödinger equation that is the basis for non-relativistic quantum mechanics, in which case $\Phi(x,t)$ is the potential for $u(x,t)$, interpreted as a wave.

2. Methods

2.1 Looking for Solitons in a Perturbed NLS

In order to find the potential for a Gross-Pitaevskii equation (1.21) which describes perturbed solutions to the Fithugh-Nagumo model, we assume solitons in the form of

$$u(x,t) = e^{i\phi} r(x,t)$$

where $\phi = bx + ct$ with b and c as constants [5]. We thus obtain the following:

$$\begin{aligned} u_t &= e^{i\phi}(icr + r_t) && (2.1) \\ u_x &= e^{i\phi}(ibr + r_x) && (2.2) \\ u_{xx} &= e^{i\phi}(-b^2 r + 2ibr_x + r_{xx}). && (2.3) \end{aligned}$$

We substitute u_t, u_x, and u_{xx} into Equation (1.21) and obtain

$$-cr - 2ikr_t = -b^2 r + 2ibr_x + r_{xx} + F(r)r + \Phi(x,t)r, \tag{2.4}$$

where $F(r) = 2r^2$. We assume that $r = f(x - 2kt)$ is a traveling wave with $2k$ being the velocity, from which it follows that

$$r_t = -2kr_x \tag{2.5}$$

Substituting these values of r_t and r_x into Equation 2.4 suggests the potential in equation (1.21) is

$$\Phi(x,t) = k^4 - c - \frac{r_{xx}}{r} - F(r). \tag{2.6}$$

Knowing what form of the potential to add to the NLS, we solve for the perturbed solutions of the Fitzhugh Nagumo model. Since our solution accounts for only the fast variable of the Fithugh-Nagumo Model, it has infinite energy. To model the fast/slow interaction, we insert a perturbation term $e^{-\delta x}$ for $\delta \approx 0$. This perturbation leads to finite energy solutions. For $a = 1$, our perturbed solutions are traveling waves of the form

$$r(x,t) = f(x-2kt) = \frac{e^{-\delta(x-2kt)}}{1 + Pe^{-(x-2kt)}}. \tag{2.7}$$

This, in turn, implies a potential of the form

$$\Phi(x,t) = M|u|u \tag{2.8}$$

where M is constant. Thus,

$$u(x,t) = e^{i\phi} r(x,t) = e^{i\phi} \frac{e^{-\delta(x-2kt)}}{1 + Pe^{-(x-2kt)}} \tag{2.9}$$

is an approximate solution to

$$i\frac{\partial u}{\partial t} = -\frac{\partial^2 u}{\partial x^2} + 2|u|^2 u - M|u|u. \tag{2.10}$$

Because (2.9) is only approximately a solution to (2.10), we now derive a numerical approximation method for (2.10).

2.2 The Pseudo-spectral Method

A pseudo-spectral method is used to numerically solve the Gross-Pitaevskii Equation [15]. The method is based on the Fourier transform. If f is continuous on $\mathbb{R}$, $\int_{-\infty}^{\infty} |f| < \infty$ and $\int_{-\infty}^{\infty} |f|^2 < \infty$, then the Fourier Transform exists and is given by

$$\mathfrak{F}(f) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i\omega x}dx. \tag{2.11}$$

It can be shown that

$$\mathfrak{F}\left(\frac{\partial f}{\partial x}\right) = 2\pi i\omega\mathfrak{F}(f). \tag{2.12}$$

Therefore,

$$\frac{\partial f}{\partial x} = \mathfrak{F}^{-1}\left(2\pi i\omega\mathfrak{F}(f)\right). \tag{2.13}$$

Also, it follows that

$$\frac{\partial^2 f}{\partial x^2} = \mathfrak{F}^{-1}\left(-4\pi\omega^2\mathfrak{F}(f)\right). \tag{2.14}$$

The pseudo-spectral method utilizes special properties of the Fourier transform and its inverses in order to solve the partial differential equation. Beginning with our perturbed NLS,

$$i\frac{\partial u}{\partial t} = -\frac{\partial^2 u}{\partial x^2} + 2|u|^2u - M|u|u, \tag{2.15}$$

Equation (2.15) is transformed into

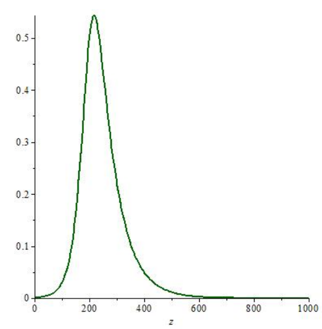
$$i\frac{\partial u}{\partial t} = \mathfrak{F}^{-1}\left(-4\pi\omega^2\mathfrak{F}(u)\right) + 2|u|^2u - M|u|u. \tag{2.16}$$

We use approximate solutions (2.9) with different values for δ as initial conditions for our approximation schemes. An ode solver is then applied to the resulting ordinary differential equation in order to integrate the solution over a time interval. The fast Fourier transform is used to approximate the continuous Fourier transform, implying that the numerical approximation is spatially periodic, although these periods are large. The solutions are then plotted in order to analyze the soliton-like characteristics of cardiac action potentials. The images in the following sections show a single periodic domain of these approximations.

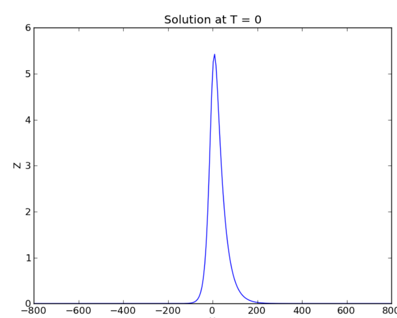
3. Results

3.1 Discussion

Figures 1 and 2 compare the calcium cardiac action potentials and solutions to our perturbed NLS at $\delta = 0.3$. Although there is a translational difference within the spatial component between the two waves in Figures 1 and 2, there is remarkable similarity of the wave shape between the solution and the actual cardiac action potential.



(a) Figure 1: Ca cardiac action potential



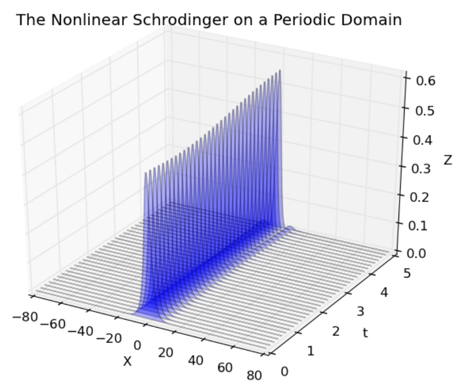
(b) Figure 2: $\delta = 0.3$, pseudo-spectral method

We let $M = 3\varepsilon$, and consider the equation

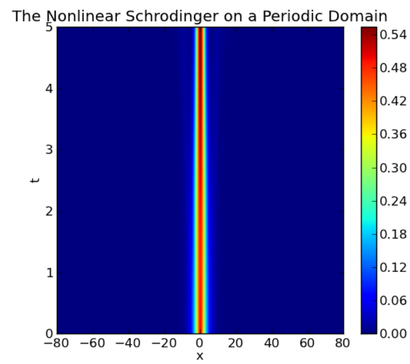
$$iU_t = -U_{xx} + 2|u|^2u - 3\varepsilon|u|u. \tag{3.1}$$

Numerical solutions to (3.1) are computed for different values of ε within the equation. Different values of δ are also chosen within the initial condition of the approximate solutions (2.9).

Figures 3 and 4 are two different visualizations of the same soliton. The plot used in Figure 4 is what will be utilized for the rest of the figures within this paper.



(c) Figure 3:$\delta = 0.5$, $\varepsilon = 1.0$



(d) Figure 4:$\delta = 0.5$, $\varepsilon = 1.0$

We generated and analyzed a large number of solutions for various initial conditions and values of the parameter ε. In particular, we explored the soliton-like properties of perturbed Fitzhugh Nagumo (pFN) solutions for single wave solutions. We also used well-separated super-positions of pairs of pFN solutions to explore interactions of individual waveforms. Here is a summary of the results for one wave:

- When $\varepsilon = 0$, a completely radiating wave is produced.
- When $\varepsilon = 1$, the wave is non-radiating
- Solutions close to $\varepsilon = 1$ have less dispersive waves.
- A value of $\delta = 0.5$ seem to create the most stable waves.

For solutions involving two waves, we have:

- The closer the value of δ is to 0.5, the less dispersive the solution becomes.

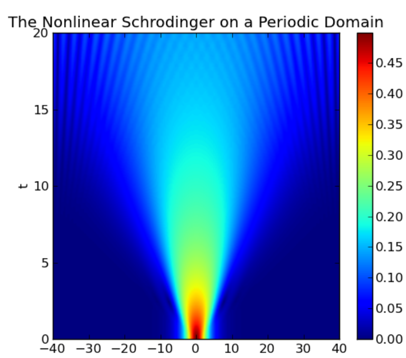
- The most stable collisions occur for values of $\varepsilon \approx 1$

- Waves with $\delta = 0.5$ have the least radiating solutions

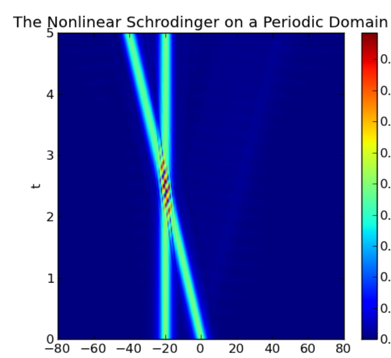
- For a value of $\delta = 0.5$, the values $\varepsilon = 0.8$, $\varepsilon = 1.0$, and $\varepsilon = 1.3$ have the least dispersive interaction between two waves.

- For small values of ε ($\varepsilon < 0.1$) the two waves tend to be stationary. This is due to the pseudo-spectral method failing to observe the collision between two waves at values less than 0.1.

Figures 5 and 6 illustrate the differences between solutions at $\varepsilon = 0$ and solutions at $\varepsilon = 1$. For $\varepsilon = 0$, waves immediately radiate as they start to travel, illustrating dispersive properties. At $\varepsilon = 1$, solutions are hardly radiating. Also, when two solutions of our perturbed NLS collide with each other at $\varepsilon = 1$, they maintain their wave-forms and only change by a slight shift in phase, behaving like solitons.

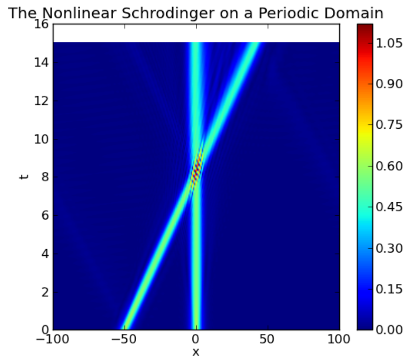


(e) Figure 5: $\varepsilon = 0$

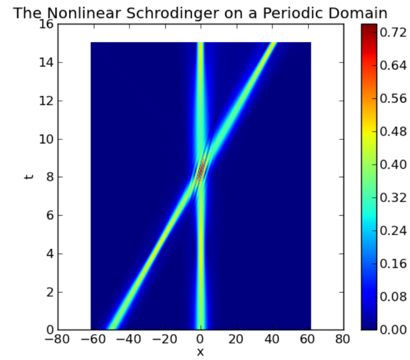


(f) Figure 6: $\varepsilon = 1$

Figures 7 and 8 show the stability of waves for values of $\varepsilon = 0.8$ and $\varepsilon = 1.3$ for a value of $\delta = 0.5$. While both waves are somewhat stable, the waves for $\varepsilon = 1.3$ are slightly less dispersive than the waves for $\varepsilon = 0.8$



(g) Figure 7: $\varepsilon = 0.8$



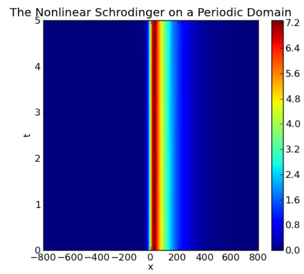
(h) Figure 8: $\varepsilon = 1.3$

The displacement between the initial and final wave are also measured. The following observations can be made about the displacement:

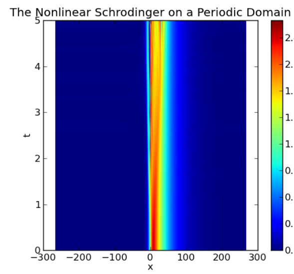
- The displacement between the initial and final wave heavily relies on the time allotted for wave propagation.
- The largest displacement occurs for values of $\varepsilon = 0.5$
- Overall, values of $\varepsilon = 0.1$ have the least amount of displacement between the initial and final wave.

The complete results for our numerical solutions are given in the next section. Results for the displacement between the initial and final wave are given in Appendix A.

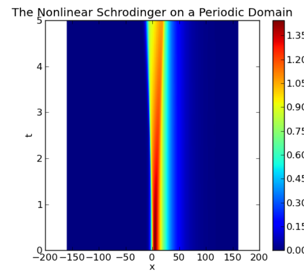
For comparison, Figure 3.1 shows particular images of solutions with initial conditions $\delta = 0.1$ for various values of ε. The values $\varepsilon = 1.0$ and $\varepsilon = 1.3$ create the least dispersive waves. It is observed no wave is created at $\varepsilon = 2.0$. This solution will be studied in more detail in the future.



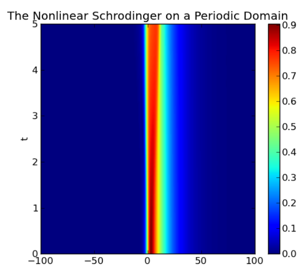
(i) $\varepsilon = 0.1$



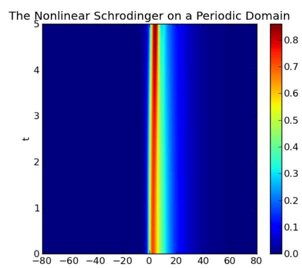
(j) $\varepsilon = 0.3$



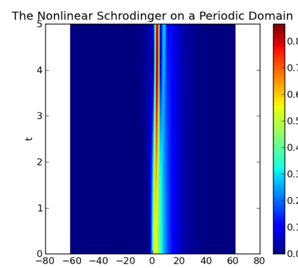
(k) $\varepsilon = 0.5$



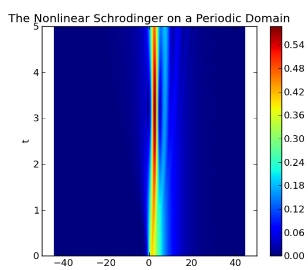
(l) $\varepsilon = 0.8$



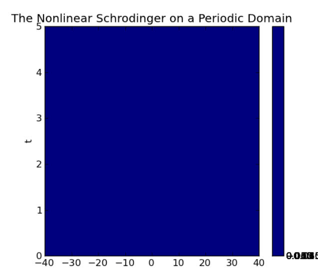
(m) $\varepsilon = 1.0$



(n) $\varepsilon = 1.3$



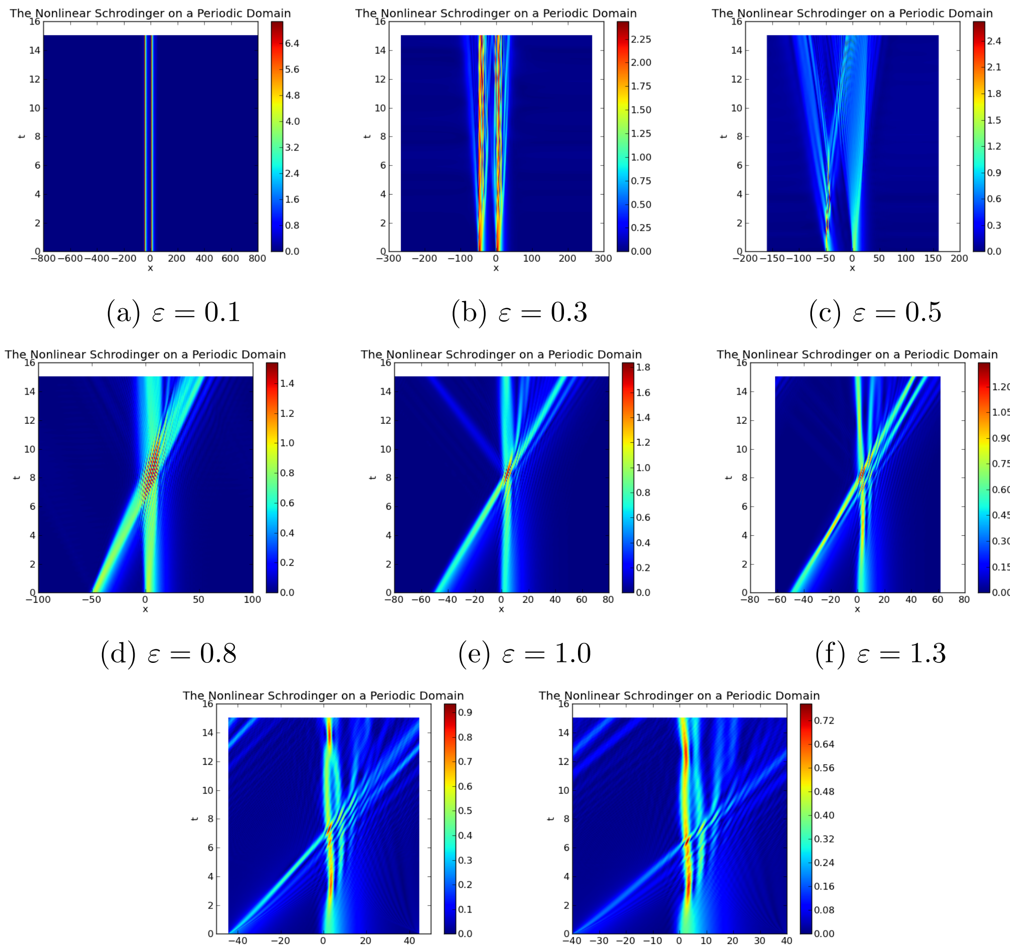
(o) $\varepsilon = 1.8$



(p) $\varepsilon = 2.0$

Figure 3.1: $\delta = 0.1$

Figure 3.2 shows images of solutions concerning two waves with initial condition $\delta = 0.1$ for various values of ε. These solutions are the most dispersive out of wave solutions for all δ values considered.



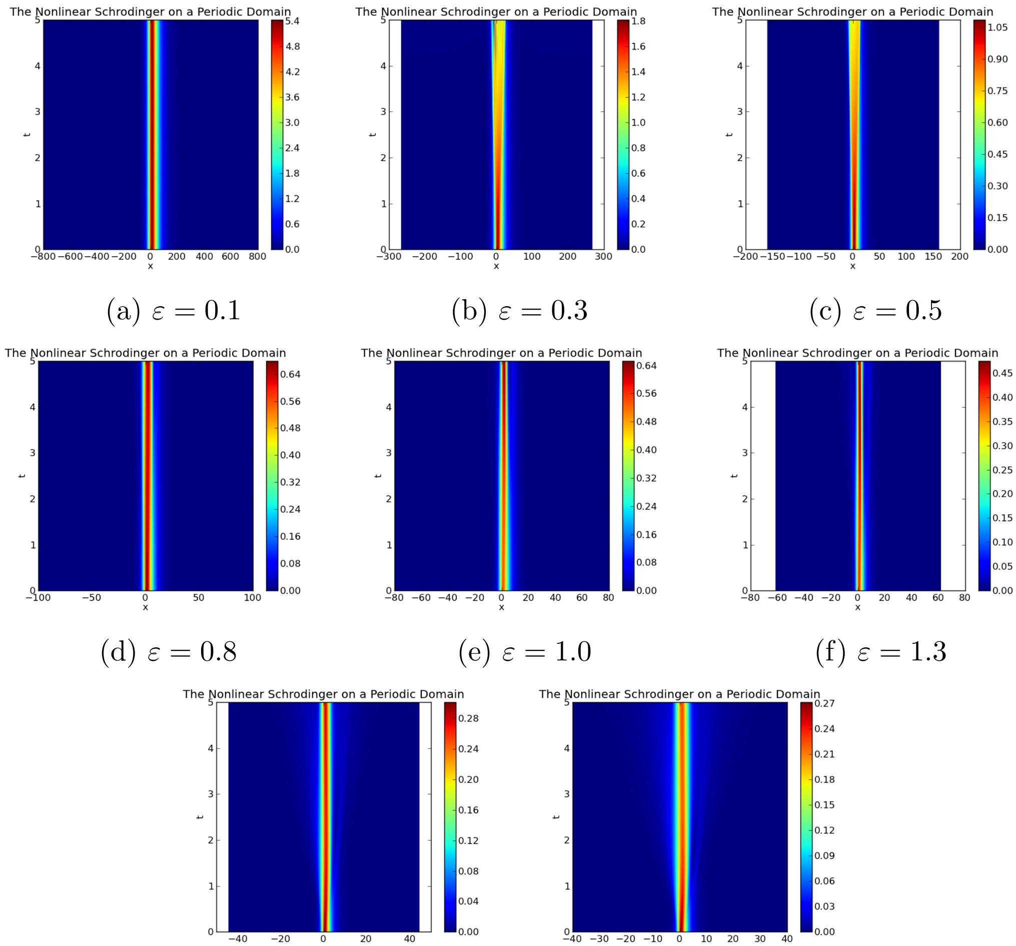
(a) $\varepsilon = 0.1$ (b) $\varepsilon = 0.3$ (c) $\varepsilon = 0.5$

(d) $\varepsilon = 0.8$ (e) $\varepsilon = 1.0$ (f) $\varepsilon = 1.3$

(g) $\varepsilon = 1.8$ (h) $\varepsilon = 2.0$

Figure 3.2: $\delta = 0.1$ two waves

For Figure 3.3, one-wave solutions with initial condition $\delta = 0.3$ for various values of ε are shown. These waves are much less dispersive than solutions with $\delta = 0.1$. These figures also illustrate that $\varepsilon = 0.8$, $\varepsilon = 1.0$, and $\varepsilon = 1.3$ are the least radiating of ε values for $\delta = 0.3$.



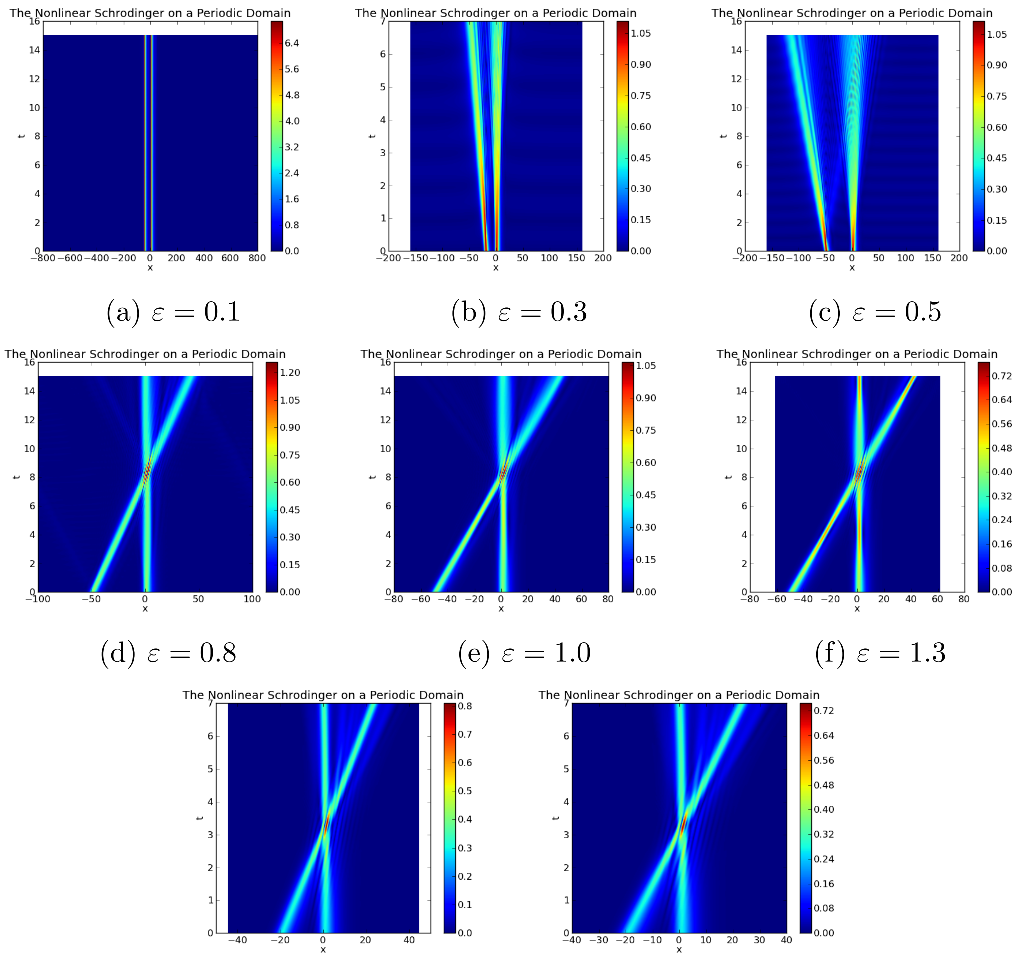
(a) $\varepsilon = 0.1$ (b) $\varepsilon = 0.3$ (c) $\varepsilon = 0.5$

(d) $\varepsilon = 0.8$ (e) $\varepsilon = 1.0$ (f) $\varepsilon = 1.3$

(g) $\varepsilon = 1.8$ (h) $\varepsilon = 2.0$

Figure 3.3: $\delta = 0.3$

Figure 3.4 shows solutions involving two waves with initial condition $\delta = 0.3$. These waves even further depict the stability of solutions at values of $\varepsilon = 0.8$, $\varepsilon = 1.0$, and $\varepsilon = 1.3$.



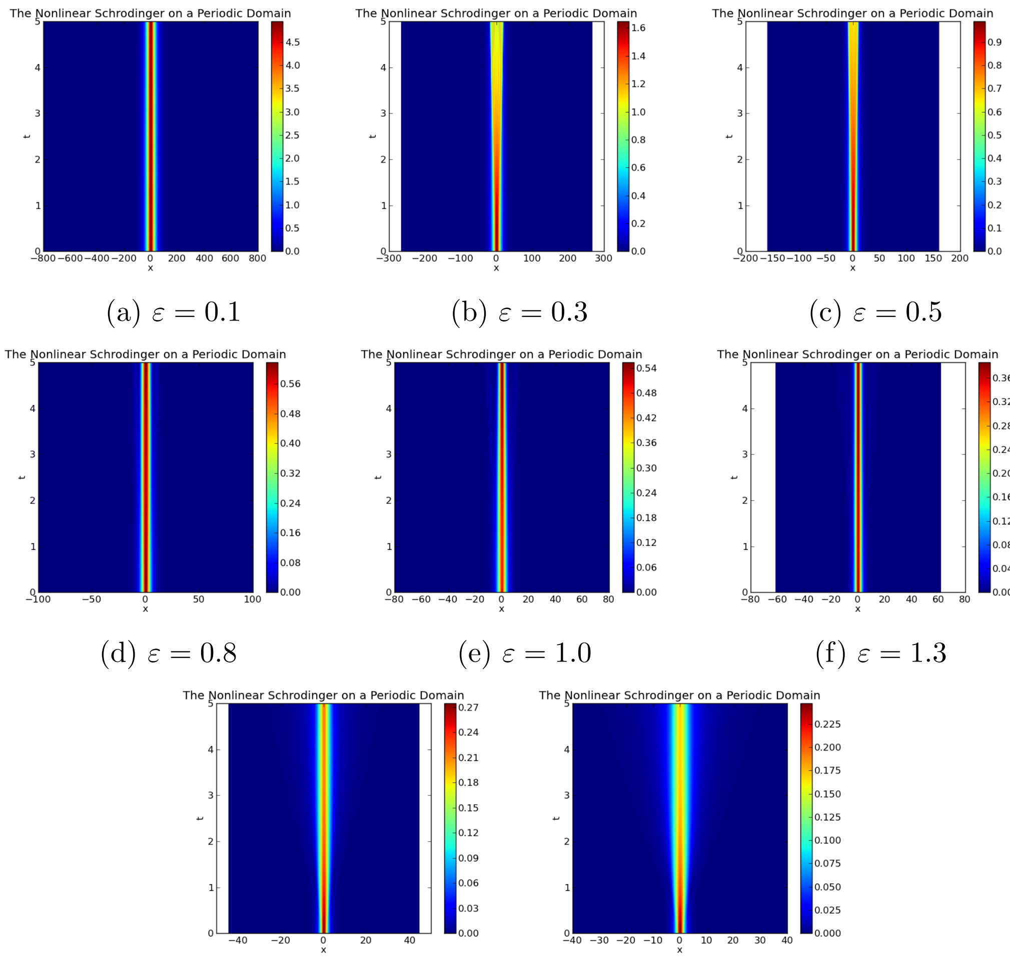
(a) $\varepsilon = 0.1$ (b) $\varepsilon = 0.3$ (c) $\varepsilon = 0.5$

(d) $\varepsilon = 0.8$ (e) $\varepsilon = 1.0$ (f) $\varepsilon = 1.3$

(g) $\varepsilon = 1.8$ (h) $\varepsilon = 2.0$

Figure 3.4: $\delta = 0.3$ two waves

Figure 3.5 shows solutions with initial condition $\delta = 0.5$ for various values of ε. These waves are the least radiating wave solutions for all δ values considered.



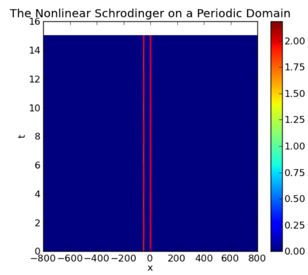
(a) $\varepsilon = 0.1$ (b) $\varepsilon = 0.3$ (c) $\varepsilon = 0.5$

(d) $\varepsilon = 0.8$ (e) $\varepsilon = 1.0$ (f) $\varepsilon = 1.3$

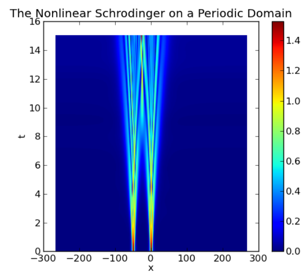
(g) $\varepsilon = 1.8$ (h) $\varepsilon = 2.0$

Figure 3.5: $\delta = 0.5$

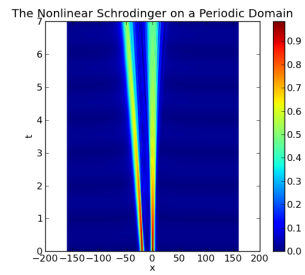
The two-wave solutions with initial condition $\delta = 0.5$ for various values of ε are illustrated in Figure 3.6. Like the one-wave solutions, these are also the least radiating waves for all δ values considered.



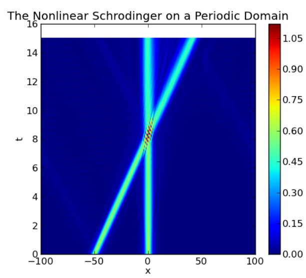
(a) $\varepsilon = 0.1$



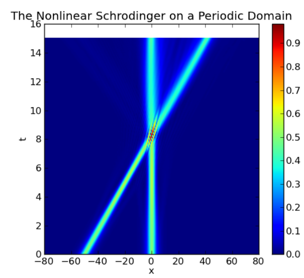
(b) $\varepsilon = 0.3$



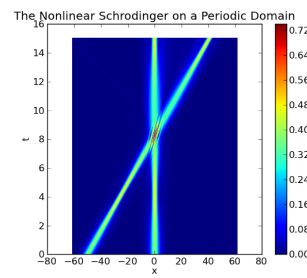
(c) $\varepsilon = 0.5$



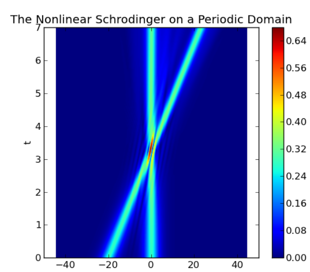
(d) $\varepsilon = 0.8$



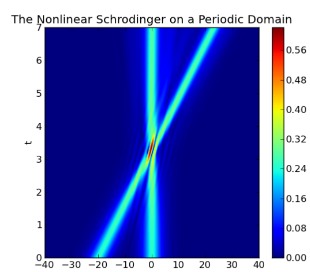
(e) $\varepsilon = 1.0$



(f) $\varepsilon = 1.3$



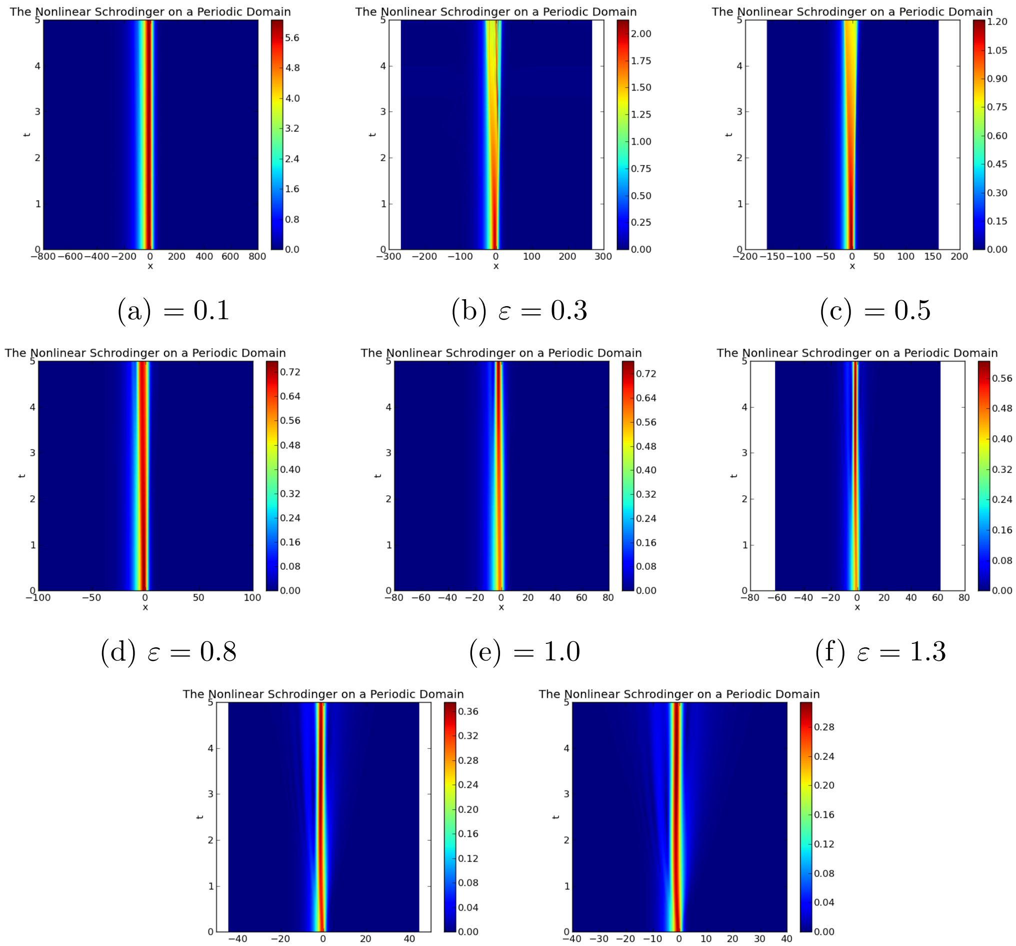
(g) $\varepsilon = 1.8$



(h) $\varepsilon = 2.0$

Figure 3.6: $\delta = 0.5$, two waves

The last δ value we observe is $\delta = 0.8$. These waves are also somewhat non-dispersive.



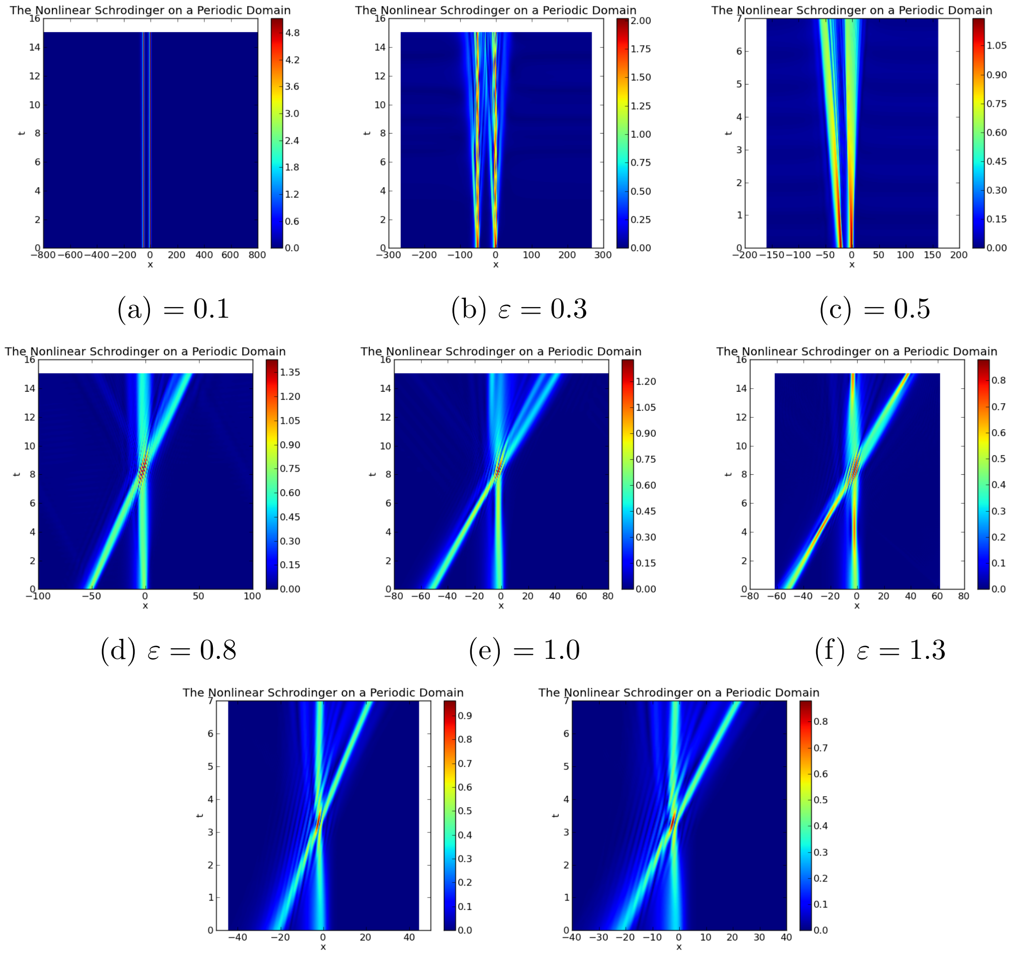
(a) $= 0.1$ (b) $\varepsilon = 0.3$ (c) $= 0.5$

(d) $\varepsilon = 0.8$ (e) $= 1.0$ (f) $\varepsilon = 1.3$

(g) $\varepsilon = 1.8$ (h) $\varepsilon = 2.0$

Figure 3.7: $\delta = 0.8$

Finally, Figure 3.8 depicts the two-wave solutions for $\delta = 0.8$. Although the one-wave solutions did not show significant radiation, the two-wave solutions illustrate the dispersiveness of solutions at this particular δ value.



(a) $= 0.1$ (b) $\varepsilon = 0.3$ (c) $= 0.5$

(d) $\varepsilon = 0.8$ (e) $= 1.0$ (f) $\varepsilon = 1.3$

(g) $\varepsilon = 1.8$ (h) $\varepsilon = 2.0$

Figure 3.8: $\delta = 0.8$, two waves

4. Symmetry Properties of The Perturbed Nonlinear Schrödinger Equation

The solutions we have found numerically complement several analytic results we have obtained concerning soliton-like properties of cardiac action potentials. There are two motives for studying the group theory of our perturbed NLS equation. First, the group theory allows us to generate initial conditions and allows us to compare them to numerical approximations. Second, the group theory implies matrix relationships known as Lax Pairs that can be used to solve an NLS equation. The solutions we have found can be extended to larger family of solutions by using Lie symmetry groups: subgroups of the permutation groups of the solutions that form smooth manifolds [16, 17]. In the next section we will introduce the ideas that make it possible to find these symmetries.

4.1 Groups, Lie Groups and Group Symmetries

We will first start with defining a Group [19]:

Definition 4.1. A group $\langle G, * \rangle$ is a set G, closed under a binary operation $*$, such that the following axioms are satisfied:

- For all a, b, c within G, we have $(a * b) * c = a * (b * c)$.
- There is an element e in G such that for all x in G, $e * x = x * e = x$.
- Corresponding to each a within G, there is an element a' in G such that $a * a' = a' * a = e$.

A group consists of a set of elements in which the above is true when the particular binary operation $*$ is induced onto the elements. The following are examples of groups:

Example 4.2. The group $O(n)$ is the set of all orthogonal matrices [20],

Example 4.3. The group $SO(n)$ is a subgroup of $O(n)$ consisting of orthogonal matrices with determinant 1 [20].

Example 4.4. The matrix group $U(n)$ is the set of all unitary matrices A. A complex matrix A is unitary if $AA^{\dagger} = A^{\dagger}A = I$ where I is the identity matrix [20].

Example 4.5. The group $SU(n)$ is a subgroup of $U(n)$ consisting of all unitary matrices with determinant 1 [19].

Another example of a group is the dihedral group of a square. The dihedral group consists of all ways in which one can rotate or flip a square and obtain the same square. For instance, one can rotate the square ninety degrees and not change the shape of the square. Since the shape of the square is unchanged, the transformations rotation and reflection can be considered symmetries of a square, and are considered to be elements in a square's group of symmetries.

Definition 4.6. The Symmetry group of a set X is the set of all 1-1 transformations of X onto itself [19].

For our perturbed NLS, we are interested in the symmetries of differential equations. These symmetries will lead us to find more solutions to our nonlinear partial differential equation, and therefore help us in the future find more solutions describing cardiac action potentials. To observe these symmetries, we look into particular symmetries groups called Lie Groups. Before introducing Lie Groups, however, we must look into the concept of a manifold.

Definition 4.7. An n-dimensional manifold is a set of points M together with a notion of open sets such that each point p within M is contained in an open set U that has a continuous bijection $\phi : U \to \phi(U) \subset \mathbb{R}^n$, where a bijection is a map that is both one-to-one and onto. If the functions ϕ are differentiable, then the manifold is considered to be a smooth manifold [20].

Therefore, an n-dimensional manifold is a space where small regions resemble $\mathbb{R}^n$. Examples of manifolds include lines (one-dimensional manifolds) and spheres (two-dimensional manifolds). Solutions to differential equation can also be interpreted as manifolds. For example, ordinary differential equation are often interpreted as curves and partial differential equations in two variables are interpreted as surfaces.

Thus, finding symmetries of a differential equation helps us determine the characteristics of the manifolds corresponding to solutions to a differential equation. Let us look at a specific subgroup of the special orthogonal group $SO(3)$ consisting of rotations about the z-axis. We can write the matrix of these rotations as such [20]:

$$T_z(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{4.1}$$

This group is, in fact, an example of a Lie group.

Definition 4.8. A Lie group is an algebraic group that forms a differentiable manifold in which the group operation is also differentiable.

To find symmetries within our perturbed NLS equation, we can check to see which Lie groups do not alter our perturbed NLS equation. In the next section, we show the symmetries we have found for our perturbed NLS equation.

4.2 Symmetries of our Perturbed NLS

To show that a partial differential equation is invariant over a Lie group, we often write the groups in the form

$$t \rightarrow f_1(t,x,u) \tag{4.2}$$

$$x \rightarrow f_2(t,x,u) \tag{4.3}$$

$$u \rightarrow f_3(t,x,u), \tag{4.4}$$

where $f_j, j = 1,2,3$ show the action of the group on a particular manifold. We have shown that (1.21) is invariant under the following groups:

$$t \rightarrow t+t_0,\ x \rightarrow x,\ u \rightarrow u.\ (\textit{time translation}) \tag{4.5}$$

$$t \rightarrow t,\ x \rightarrow x+x_0,\ u \rightarrow u.\ (\textit{spatial translation}) \tag{4.6}$$

$$t \rightarrow t,\ x \rightarrow x-ct,\ u \rightarrow ue^{i\frac{c}{2}(x-\frac{c}{2}t)}.\ (\textit{Galilean invariance}) \tag{4.7}$$

For example, substituting (4.4) into

$$i\frac{\partial u}{\partial t} = -\frac{\partial^2 u}{\partial x^2} + 2|u|^2u - M|u|u \tag{4.8}$$

leads to

$$i\frac{\partial (ue^{i\frac{c}{2}(x-\frac{c}{2}t)})}{\partial t} = -\frac{\partial^2 (ue^{i\frac{c}{2}(x-\frac{c}{2}t)})}{\partial (x-ct)^2} + 2|ue^{i\frac{c}{2}(x-\frac{c}{2}t)}|^2ue^{i\frac{c}{2}(x-\frac{c}{2}t)} - M|ue^{i\frac{c}{2}(x-\frac{c}{2}t)}|ue^{i\frac{c}{2}(x-\frac{c}{2}t)}. \tag{4.9}$$

However $|ue^{i\frac{c}{2}(x-\frac{c}{2}t)}| = |u|$. Thus, we can simplify the equation to the following:

$$e^{i\frac{c}{2}(x-\frac{c}{2}t)}i\frac{\partial u}{\partial t} = e^{i\frac{c}{2}(x-\frac{c}{2}t)}\left(-\frac{\partial^2 u}{\partial x^2} + 2|u|^2u - M|u|u\right) \tag{4.10}$$

The exponential terms cancel. Therefore, the perturbed NLS is Galilean invariant. Determining spatial and temporal symmetries follow the same procedure.

4.3 Inverse Scattering Transform for the NLS

In many Lie groups, the multiplicative order is important. This leads to the following definition [20].

Definition 4.9. A binary operation is commutative (equivalently abelian) if $a * b = b * a$ for all a,b within a group G.

For example, SO(3) is a non-abelian group. If a Lie Group G is non-abelian, then the tangent space to the identity (G as a manifold) is the Lie Algebra of G, denoted g. A Lie algebra is a vector space of linear transformations, and if $a, b \in g$, then $[a, b] = ab - ba$ is the Lie bracket on g. The Lie bracket is derived from the group operation on G.

The Lie algebra of a symmetry group of a nonlinear PDE can sometimes be used to obtain solutions to the PDE. For example, many PDE's, including the KdV and the NLS, can be solved using the method of Lax pairs. A Lax pair consists of two matrices A and B that satisfy the following equation:

$$A_t = B_x + [B, A]. \tag{4.11}$$

The method for discovering the solutions to a complex partial differential equation is to postulate a given A and B, and determine which partial differential equation is the compatibility equation for the two matrices. Typically, the choice of A and B is of the partial differential equation's symmetry group dictated by the Lie algebra. We demonstrate this method with the nonlinear Schrödinger equation [5, 21]. If σ_z is the element of the Lie algebra $SU(2)$ that generates rotations about the z axis,

$$L = -i\sigma_z \frac{\partial}{\partial x} + \begin{pmatrix} 0 & u \\ -u^* & 0 \end{pmatrix} \tag{4.12}$$

By taking into account the Lax eigenvalue problem $L\phi = \xi\phi$, we can obtain the first Lax Pair equation for the NLS shown previously:

$$Y_x = \begin{pmatrix} -i\xi & u \\ -u^* & i\xi \end{pmatrix} Y. \tag{4.13}$$

We then let $Y_x = AY$ and $Y_t = BY$. Our goal is to find B so that $Y_{xt} = Y_{tx}$. By finding this matrix, one can in the process obtain the compatibility equation in which $Y_{xt} = Y_{tx}$ if and only if u is a solution to the specified equation. We first take the time derivative of Y_x and the spatial derivative of Y_t:

$$Y_{xt} = A_tY + AY_t = A_tY + ABY \tag{4.14}$$

$$Y_{tx} = B_xY + BY_x = B_xY + BAY \tag{4.15}$$

The condition $Y_{xt} = Y_{tx}$ then implies

$$A_tY + ABY = B_xY + BAY, A_t = B_x + [B, A]. \tag{4.16}$$

To dive further into finding the matrix B, we take into account that the matrix is within the group $SU(2)$, giving us B in the form of the lie algebra $SU(2)$:

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}. \tag{4.17}$$

Thus,

$$AB = \begin{pmatrix} -\bar{b}u - ia\xi & u\bar{a} - ib\xi \\ -i\bar{b}\xi - au^* & i\bar{a}\xi - bu^* \end{pmatrix} \tag{4.18}$$

$$BA = \begin{pmatrix} -ia\xi - bu^* & au + ib\xi \\ i\bar{b}\xi - \bar{a}u^* & -\bar{b}u + i\bar{a}\xi \end{pmatrix}. \tag{4.19}$$

Therefore,

$$[B, A] = \begin{pmatrix} \bar{b}u - bu^* & 2ib\xi + (a - \bar{a})u \\ 2i\bar{b}\xi + (a - \bar{a})u^* & bu^* - \bar{b}u \end{pmatrix}. \tag{4.20}$$

Our equation $A_t = B_x + [B, A]$ then becomes

$$\begin{pmatrix} 0 & u_t \\ -u_t^* & 0 \end{pmatrix} = \begin{pmatrix} a_x & b_x \\ -\bar{b}_x & \bar{a}_x \end{pmatrix} + \begin{pmatrix} \bar{b}u - bu^* & 2ib\xi + (a - \bar{a})u \\ 2i\bar{b}\xi + (a - \bar{a})u^* & bu^* - \bar{b}u \end{pmatrix}. \tag{4.21}$$

We arrive at the following equations:

$$a_x = bu^* - \bar{b}u \tag{4.22}$$

$$u_t = b_x + 2ib\xi + (a - \bar{a})u \tag{4.23}$$

$$\bar{a}_x = \bar{b}u - bu^* \tag{4.24}$$

$$-u_t^* = -\bar{b}_x + 2i\bar{b}\xi + (a - \bar{a})u^*. \tag{4.25}$$

The last two equations are simply conjugates of the first two equations. Our next step is to find $a(x,t)$ and $b(x,t)$ so that these equations are true [5]. To do so, we assume that

$$a = a_0 + a_1\xi + a_2\xi^2 \tag{4.26}$$

$$b = b_0 + b_1\xi + b_2\xi^2. \tag{4.27}$$

By inserting these conditions for the functions a and b, we arrive at the following for the first two equations:

$$\begin{aligned} a_{0,x} + a_{1,x}\xi + a_{2,x}\xi^2 &= b_0u^* - b_0^*u + (b_1u^* - b_1^*u)\xi + (b_2u^* - b_2^*u)\xi^2 \\ u_t &= b_{0,x} + b_{1,x}\xi + b_{2,x}\xi^2 + 2i(b_0\xi + b_1\xi^2 + b_2\xi^3) \\ &+ (a_0 - a_0^*u + (a_1 - a_1^*)u\xi + (a_2 - a_2^*)u\xi^2. \end{aligned}$$

Upon collecting coefficients and observing that $b_2 = 0$ and $b_{2,x} = 0$, one arrives at

$$u_t = b_{0,xu}(a_0 - a_0^*) \tag{4.28}$$

$$-2ib_0 = b_{1,x} + (a_1 - a_1^*)u \tag{4.29}$$

$$-2ib_1 = (a_2 - a_2^*)u. \tag{4.30}$$

Moreover, $a_{2,x} = 0$, which implies that a_2 is constant. Setting this as $-2i$ gives us $b_0 = iu_x$. We next observe $a_0 = i|u|^2$ since $a_{0,x} = \frac{\partial}{\partial_x}(iu^*u)$. Our last observation considers $u_t - b_{0,xu}(a_0 - a_0^*)$. Substituting the previous values for $b_{0,x}$ and a_0 produces the nonlinear Schrödinger equation:

$$iu_t = -u_{xx} - 2|u|^2u, \tag{4.31}$$

Thus, for the final solutions $a(x,t)$ and $b(x,t)$, the nonlinear Schrödinger equation is the compatibility equation for the Lax Pair (4.32).

In order to find a family of solutions to our perturbed NLS equation, our next step is finding the Lax Pair to which our perturbed NLS equation is the compatibility equation for. This, however, is a future direction that may require generalization to a higher dimensional setting.

4.4 The Infinite Number of Conservation Laws of the NLS equation

A critical property of the NLS equation is its infinite number of conservation laws, meeting a criterion for integrability. A conservation law states that certain physical properties of a system, such as energy, momentum, and angular momentum, do not change in time within an isolated physical system. A conservation law is in the form of $A_t + B_x = 0$ due to Noether's theorem stating that any differential symmetry of the integral of a Lagrangian function over time has a corresponding conservation law [23]. For example, if the Lagrangian of a physical system is independent of the location of the origin of the system, then the system will conserve angular momentum. The method we use to derive the infinite number of conservation laws of the NLS equation is a variation of the Zakharov-Shabat method [21]. Consider a solution $Y = (y_1, y_2)^T$ to the two Lax Pair equations:

$$Y_x = \begin{pmatrix} -i\xi & u \\ -u^* & i\xi \end{pmatrix} Y, \tag{4.32}$$

$$Y_t = \begin{pmatrix} -2i\xi^2 + i|u|^2 & iu_x + 2\xi u \\ iu_x^* - 2\xi u^* & 2i\xi^2 - i|u|^2 \end{pmatrix} Y \tag{4.33}$$

Defining $\mu = y_1/y_2$,

$$Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \qquad y_{2,x} = -u^* y_1 + i\xi y_2. \tag{4.34}$$

Dividing by y_2,

$$\frac{y_{2,x}}{y_2} = \frac{-u^* y_1}{y_2} + i\xi. \tag{4.35}$$

Thus, integrating this we arrive at the following equation:

$$\ln(y_2)_x = i\xi - u^* \mu. \tag{4.36}$$

By the same procedure we can find the following equation for $ln(y_2)_t$:

$$\ln(y_2)_t = \left(2i\xi^2 - i|u|^2\right) + \left(iu_x^* - 2\xi u^*\right)\mu. \tag{4.37}$$

Since $(Y_x)_t = (Y_t)_x$, we have

$$\left(i\xi - u^* \mu\right) = \left(2i\xi^2 - i|u|^2\right) + \left(iu_x^* - 2\xi u^*\right)\mu. \tag{4.38}$$

We can expand μ as a function of ζ as a series at ∞ :

$$\mu(x,t,\xi) = -\sum_{n=1}^{\infty} \frac{\mu_n(x,t)}{(-2i\xi)^n}. \tag{4.39}$$

Substituting this into the previous equation yields

$$-\left(u^* \mu\right) = -\sum_{n=1}^{\infty} \frac{\left(u^* \mu_n(x,t,\xi)\right)}{(-2i\xi)^n}, \tag{4.40}$$

Or equivalently,

$$-iu^* u = i \sum_{n=1}^{\infty} \frac{\left[\left(u_x^* - 2\xi u^*\right) \mu_n\right]_x}{(-2i\xi)^n}. \tag{4.41}$$

Since for $n = 1$, $\left(u^* \mu_1\right)_t = \left[(u_x^* - 2\xi u^*)\mu_n\right]_x$, we have

$$(-iu^* u)_x = \sum_{n=1}^{\infty} \frac{(iu_x^* \mu_n)_x}{(-2i\xi^n)} = \sum_{n=1}^{\infty} \frac{i(u^* \mu_n)_x}{(-2i\xi)^{n-1}} \tag{4.42}$$

Setting $j = n - 1$ in the second sum, we obtain

$$(-iu^* u)_x = \sum_{n=1}^{\infty} \frac{(iu_x^* \mu_n)_x}{(-2i\xi^n)} = \sum_{n=1}^{\infty} \frac{i(u_* \mu_{j+1})_x}{(-2i\xi)^j}. \tag{4.43}$$

Taking out the $n = 1$ term from the series then leads to

$$(-iu^*(u - \mu_1))_x + \sum_{n=1}^{\infty} \frac{(-iu_x^*\mu_n)_x + u^*\mu_{n+1}}{(-2i\xi)^n} = \sum_{n=1}^{\infty} \frac{(u^*\mu_{n+1})_t}{(-2i\xi)^n}. \tag{4.44}$$

For (4.46) to be satisfied, we first must have that $\mu_1 = u$.Under the same concept, $\mu_u = u_x$, and so on, leading to

$$(u^*\mu_n)_t = i\left(u^*\mu_{n+1} - u_x^*\mu_n\right)_x. \tag{4.45}$$

These equations give the infinite number of conservation laws for the nonlinear Schrödinger equation, with $u^*\mu_n$ being the density and $u^*\mu_{n+1} - u_x^*\mu_n$ being the flux. Thus we have the infinite number of conserved quantities as:

$$I_n = \int_{-\infty}^{\infty} u^*\mu_n dx. \tag{4.46}$$

It can be shown that this is all the integrals (conserved quantities) of the partial differential equation, and thus, the NLS is a completely integrable system.

5. Future Work

Analyzing the soliton-like properties of cardiac action potentials numerically is still in process. Statistical information such as confidence intervals, standard deviations and variances of the results from numerical methods will soon be calculated. Due to errors within the pseudalspectral method, a new computational method will be adopted in the near future. Also, the family of solutions describing cardiac action potentials will be discovered by working with more Lie symmetry groups and the Lax pair of our perturbed NLS equation.

There is also value in extending these results to higher spatial dimensions. Although not all the methods for working with one dimensional NLS equations translates into higher dimensional NLS contexts, some of the results in this paper should extend to higher dimensional settings. Thus, we will also explore higher dimensional, cardiac action potential-like solitons in the near future. We will study the behavior of these waves by extending the symmetry group of the perturbed NLS. We will also look into other types of solutions to our NLS equation, such as the possibility of breather (i.e. rogue wave) solutions.

Moreover, there are a number of applications to neuroscience not even considered. For example, our modified NLS suggests a modified Fitzhugh-Nagumo equation. In particular, the soliton approach and symmetry groups might reveal valuable information about the interplay of the fast and slow signals that make up cardiac action potentials.

6. Appendices

6.1 Appendix A: Displacement Tables

The tables below give the amount of displacement between the initial and final waves within the solution at different values of δ and ε. While δ is the perturbation to take into account the slow variable of the Fitzhugh-Nagumo equation, the ε value is the perturbation for our Gross-Pitaevskii equation. Values of $\delta = 0.1, 0.3, 0.5$, and 0.8 are measured. Values of $\varepsilon = 0.1, 0.3, 0.5, 0.8, 1.0, 1.3, 1.8$, and 2.0 are observed. As previously stated, the largest displacement occurs for values of $\varepsilon = 0.5$ while values of $\varepsilon = 0.1$ have the least amount of displacement between the initial and final wave.

One Wave		
δ	ε	x
0.1	0.1	0.047797
	0.3	0.041067
	0.5	0.364489
	0.8	0.133408
	1.0	0.137228
	1.3	0.276590
	1.8	0.071318
	2.0	*No Wave*
0.3	0.1	0.035201
	0.3	0.098601
	0.5	0.335753
	0.8	0.017619
	1.0	0.109503
	1.3	0.048222
	1.8	0.024300
	2.0	0.044155

One Wave (Cont.)		
δ	ε	x
0.5	0.1	0.002190
	0.3	0.482510
	0.5	0.318628
	0.8	0.001088
	1.0	0.059512
	1.3	0.005431
	1.8	0.061100
	2.0	0.782587
0.8	0.1	0.025733
	0.3	0.334300
	0.5	0.354331
	0.8	0.060434
	1.0	0.157177
	1.3	0.116652
	1.8	0.019820
	2.0	0.003522

Two Waves		
δ	ε	x
0.1	0.1	0.001698
	0.3	0.257342
	0.5	0.312770
	0.8	0.219724
	1.0	0.058462
	1.3	0.224882
	1.8	0.165259
	2.0	0.125386
0.3	0.1	0.003597
	0.3	0.255729
	0.5	0.295115
	0.8	0.136808
	1.0	0.107320
	1.3	0.160305
	1.8	0.055911
	2.0	0.047498

Two Waves (Cont.)		
δ	ε	x
0.5	0.1	0.093370
	0.3	0.057261
	0.5	0.107774
	0.8	0.140350
	1.0	0.056337
	1.3	0.107774
	1.8	0.032499
	2.0	0.028547
0.8	0.1	0.001189
	0.3	0.368053
	0.5	0.494624
	0.8	0.166655
	1.0	0.130141
	1.3	0.245895
	1.8	0.113831
	2.0	0.112991

6.2 Appendix B: Code

The following is our code utilizing the pseudo-spectral method in order to solve our perturbed nonlinear Schrödinger equation. The code was adapted from a Scipy Cookbook KdV example.

```
import numpy as np
from scipy.integrate import odeint
from scipy.fftpack import diff as psdiff

#from mpl_toolkits.mplot3d import Axes3D
from matplotlib.collections import PolyCollection
from matplotlib.colors import colorConverter

#from mpl_toolkits.mplot3d import axes3d
import matplotlib.pyplot as plt



def shr_exact(x, c):
    """Profile of the exact solution to the KdV for a single soliton
    on the real line."""
    #u = 1.2*1/(np.cosh(1.2*(x+20)))+np.exp(8j*(x))*0.8*1/(np.cosh(.8*x)
    eps = 1.0

    delta = 0.8

    beta = eps
```

```
    gamma = 1/eps
    u = 1.2*(1/(np.cosh(x+50)))*np.exp(1j*x)+0.8*(1/np.cosh(0.8*x))
   # u =( np.exp(-delta*x))/(1+np.exp(-x))+(np.exp(-delta*(x+20)))/(1+np
   # u = np.exp(-delta*(beta*(x)))*np.exp(0j*(x))/(1+np.exp(-(beta*(x)))
   #u = np.exp(-delta*(beta*x))/(1+np.exp(-(beta*x)))
    #u = gamma*u
    u = np.array(u, dtype=np.complex64)
    u = np.array([u.real,u.imag])
    u = u.flatten()

    return u

def shr(u, t, L):
    """Differential equations for the KdV equation, discretized in x."""
    # Compute the x derivatives using the pseudo-spectral method.
   # ux = psdiff(u, period=L)
    #eps = 1.0
    #gamma = 1/eps
    n = len(u)
    uxxRe = psdiff(u[0:(n/2)], period=L, order=2)
    uxxIm = psdiff(u[(n/2):n], period=L, order=2)

    uxx = np.array([uxxRe,uxxIm])
    uxx = uxx.flatten()

    absu =np.sqrt(u[0:n/2]**2+u[n/2:n]**2)
```

```
    absu = np.array([absu,absu])
    absu = absu.flatten()

    absu2 = u[0:n/2]**2+u[n/2:n]**2
    absu2 = np.array([absu2,absu2])
    absu2 = absu2.flatten()

    # Compute du/dt = -i*( -uxx - 2abs(u)u ) = i * (uxx + 2abs(u)u)
    dudt = (-1*2*absu2)*u + uxx + eps*(3*absu)*u
    idudt= np.array([-1*dudt[(n/2):n], dudt[0:(n/2)]])
    return idudt.flatten()
    #return ( idudt.real, idudt.imag )


# Set the size of the domain, and create the discretized grid.
eps = 1.0
beta = eps
L =160.0/beta
N = 256
dx = L/N
x = np.linspace(-L/2, L/2, N)
x1 = np.linspace(-L/beta, L/beta, N)

# Set the initial conditions.
# Not exact for two solitons on a periodic domain, but close enough...
u0 = shr_exact(x, 0.75) # +  kdv_exact(x-0.65*L, 0.4)
```

```
# Set the time sample grid.
#ps = .01
#alpha = eps**2
Tm = 7
t = np.linspace(0, Tm, 1000)



#t = alpha*t


print "Computing the solution."
from mpl_toolkits.mplot3d import Axes3D
from matplotlib.collections import PolyCollection
from matplotlib.colors import colorConverter
sol = odeint(shr, u0, t, args=(L,), mxstep=500)
sol = sol[:, 0:N] + 1j *sol[:,N:(2*N) ]



print "IMshow."


plt.figure(figsize=(6,5))
plt.imshow(np.abs(sol[::-1, :]), extent=[-L/2,L/2,0,Tm])
plt.colorbar()
plt.xlabel('x')
plt.ylabel('t')
plt.axis('normal')
```

```
plt.title('The Nonlinear Schrodinger on a Periodic Domain')
#plt.show()

#print "Wireframe."

#fig = plt.figure()
#ax = fig.add_subplot(111, projection='3d')
#tind = range(0,len(t),10)
#xind = range(0,len(x),5)
#tt = t[tind]
#xx = x[xind]
#ux = abs(sol)[:,xind]
#uu = ux[tind,:]
#X,T = np.meshgrid(xx,tt)
#ax.plot_wireframe(X,T,uu )

#plt.show()

print("WaterFall.")

## Redo the sampling
tind = range(0,len(t),30)
xind = range(0,len(x),1)
tt = t[tind]
xx = x[xind]
ux = abs(sol)[:,xind]
```

```
#The figure
fig = plt.figure()
ax = fig.gca(projection='3d')

cc = lambda arg: colorConverter.to_rgba(arg, alpha=0.6)

verts = []
for i in tind:
    verts.append( zip(xx,ux[i,:]) )



poly = PolyCollection(verts, facecolors = [cc('b')])
poly.set_alpha(0.3)
ax.add_collection3d(poly, zs=tt, zdir='y')

ax.set_xlabel('X')
ax.set_xlim3d(-L/2,L/2)
ax.set_ylabel('t')
ax.set_ylim3d(0,Tm)
ax.set_zlabel('Z')
ax.set_zlim3d(0,1.1*np.max(abs(sol)))
plt.title('The Nonlinear Schrodinger on a Periodic Domain')
plt.show()

plt.figure()
```

```
plt.plot(xx, abs(sol[0]))
plt.xlabel('X')
plt.ylabel('Z')
plt.title('Solution at T = 0')
plt.show()

plt.figure()
plt.plot(xx, abs(sol[999]))
plt.xlabel('X')
plt.ylabel('Z')
plt.title('Solution at Max Time')
plt.show()

Diff = np.max(abs(sol[0]))-np.max(abs(sol[999]))
Diff = abs(Diff)
print Diff
```

Bibliography

[1] Infeld and Rowlands, G.: Nonlinear Waves, Solitons, and Chaos. In: New York,New York E. (1990)

[2] Hasegawa, A.; Tappert, F.: Transmission of stationary nonlinear optical pulses in dispersive dielectric fibers. I. Anomalous dispersion. Appl. Phy. Lett. 23 (3): 142-144(1973). doi:10.1063/1.1654836

[3] Sinkala, Z.:Soliton/exciton transport in proteins. Theor. Biol.241 (4): 91927 (2006).doi:10.1016/j.jtbi.2006.01.028.

[4] Yakushevich, Ludmila V.: Nonlinear physics of DNA (2nd revised ed.). In: Garching, Germany (2004).

[5] Grimshaw,R., and Khusnutdinova, K.: Nonlinear Waves. (2011)

[6] Brooks, J.: A Singular Perturbation Approach to the Fitzhugh-Nagumo PDE for Modeling Cardiac Action Potentials. Masters thesis, East Tennessee State University. E-thesis-http://libraries.etsu.edu/record=b2340298 S1a(2011).

[7] "About Arrhymia."'(2011).American Heart Association.Retrieved February 24,2013 from http://www.heart.org/HEARTORG/Conditions/Arrhythmia/AboutArrhythmia/About-Arrhythmia UCM 002010 Aricle.jsp.

[8] Noble, Denis(2007).:From the Hodgkin-Huxley Axon to the Virtual Heart. *Journal of PHysiology.*Vol. 580, No.1, 15-22.

[9] Wilk, Malgosia(2007). "Physiology of Excitable Cells".Retrieved on February 24, 2013, from http://www.uta.edu/biology/wilk/classnotes/cellphy/excitable%20 cells.pdf.

[10] Kenner,J., Sneyd, J.(2004).*Mathematical Physiology.* New York: Springer Science.

[11] Klabunde, Richard E.(2008)."Non-Pacemaker Action Potentials."*Cardiovascular Physiology Concepts.*Retrieved February 24, 2013, from http://www.cvphysiology.com/Arrythmias/A006.htm.

[12] "Denis Noble."(2005) *Oxford Cardiac Electrophysiology Group.* Retrieved February 24, 2013 from http://noble.physiol.ox.ac.uk/People/DNoble.

[13] n.p. *Solitons, a Brief History of.*n.p. Retrieved February 24, 2013, from www.weizmann.ac.il/home/fnfal/papers/soliton.pdf.

[14] Odom, Robert. (2008). "First Unchanging 'Soliton' Wave Found in Space.' Retrieved February 26, 2013, from http://www.newscientist.com.

[15] Gottlieb, D. and Orzag, S.: Numerial Analysis of Spectral Methods: Theory and Applications. In: Philadelphia, PA. (2011)

[16] Gagnon,L. and Winternitz, P.: Lie Symmetries of a generalised non-linear Schrodinger equation: I. The symmetry group and its subgroups. J. Phys A. Gen. 21: 1493- 1511.(1988)

[17] Popovychh, R.O. and Eshraghi, H.: Admissible Point Transformations of Nonlinear Schrodinger Equations. pp. 167-174. (2005)

[18] n.p. (2012).*Shadows.* Retrieved February 24, 2013, from http://www.fotopedia.com/items/flickr-87691989.

[19] Fraleigh, J.B., Katz, V.J.: A First Course in Abstract Algebra. In: New Jersey.(2003)

[20] Larson, D. (2005, Augst 4). *How to talk to a Physicist: Groups, Symmetry, and Topology.*

[21] Yang, J.: Nonlinear Waves in Integrable and Nonintegrable Systems. In: Philadelphia, PA.(2010)

[22] Weckesseer, W.(2003, February 22).Cookbook/KdV.Retrieved on November 24, 2012, from http://www.scipy.org/Cookbook/KdV

[23] Noether E. (1918). "Invariante Variationsprobleme"'. Nachr. D. König. Gesellsch. D. Wiss. Zu Göttingen, Mathphys. Klasse 1918:235-257.