Mode Vertices and Mode Graphs.

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MODE VERTICES

AND MODE GRAPHS

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APPROVAL

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ABSTRACT

AN INTRODUCTION TO MODE GRAPHS

by

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The *eccentricity* of a vertex v, $\mathrm{e}_G(v)$, of a connected graph G is the distance to the farthest vertex from v, or $\max_{u \in G} \mathrm{d}(u, v)$. A *mode vertex* of a connected graph G is a vertex whose eccentricity occurs as often in the eccentricity sequence of G as the eccentricity of any other vertex. The *mode* of a graph G is the subgraph induced by the mode vertices of G. A *mode graph* is a connected graph for which each vertex is a mode vertex. An $e_1, e_2, \cdots e_k$ *mode graph* is a mode graph with eccentricities $e_1, e_2, \cdots e_k$. Note that mode graphs are a generalization of self-centered graphs. This paper presents some results based on these definitions.

DEDICATION

To our Lord and Savior, Jesus Christ. Without Him this would not have been possible. May the love and joy of God abound in the center of the universe, the periphery of the universe, and throughout the entire universe.

ACKNOWLEDGEMENTS

Thank you friends, family, and teachers. I am blessed by your love, trust, forgiveness, and wisdom. These blessings give me strength and courage to embrace life and God's plan for me.

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CHAPTER 1

INTRODUCTION

Two widely studied topics in graph theory and networking are those of distance and the related topic, centrality. Related to these topics, this paper introduces mode vertices, the mode of a graph and, in turn, mode graphs.

Definitions and theorems from [1] will be used throughout this paper. A *path* is an alternating sequence of vertices and edges such that no vertex is repeated. A graph G is said to be *connected* if for any given vertices u, $v \in V(G)$ there exists a u-v path. The *length* of a path is the number of edges in the path, and the *distance* between two vertices u and v is the length of a shortest path between u and v. The *eccentricity* of a vertex v, $\mathrm{e}_G(v)$, of a connected graph G is the distance to a furthest vertex from v, or $\max_{u \in G} \mathrm{d}(u, v)$. The *radius* of G, $rad(G)$, is the minimum of the eccentricities of G, while the *diameter* of G, $diam(G)$, is the maximum of the eccentricities of G. A vertex is a *central vertex* of G if its eccentricity is equal to the radius of G whereas a vertex is a *peripheral vertex* of G if its eccentricity is equal to the diameter of G. The *center* of a graph G, $Cen(G)$, is the subgraph induced by the central vertices of G, and the *periphery* of G, $Per(G)$, is the subgraph induced by the peripheral vertices of G. The *eccentricity sequence* of a graph G, $ES(G)$ is the sequence of eccentricities of G written in non-decreasing order.

The graph in Figure 1 illustrates several of these concepts. Two possible paths from v to z are v, vw, w, wy, y, yz, z with length 3, and v, vy, y, yz, z with length 2. The path with length 2 is the shortest path between v and z, therefore the distance between

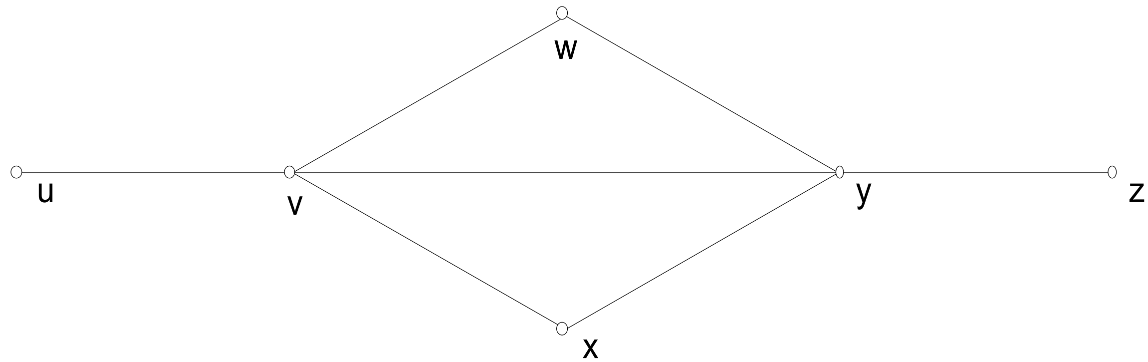

Figure 1: A connected graph G.

v and z is 2. Also, the furthest vertex from v is z, so the $\mathrm{e}(v) = 2$. The minimum of the eccentricities of the vertices of G is 2, and the maximum is 3. Therefore, the radius of G is 2, and the diameter of G is 3. The vertices v, w, x, and y all have eccentricity equal to the radius, so they are central vertices. Vertices u and z are peripheral vertices because their eccentricities are equal to the diameter. The eccentricity sequence for G is $\mathrm{ES}(G) = 2, 2, 2, 2, 3, 3$. For convenience, we write $\mathrm{ES}(G) = 2^4, 3^2$.

It may be important not only to know which vertices have certain eccentricities, but also how many vertices there are with that eccentricity. For instance, emergency facilities should be centrally located in order to reduce response time to an emergency anywhere in the network. However, if there is not enough room in the center for the emergency facilities required, or if the network is too spread out, it may be necessary to locate some emergency facilities elsewhere.

We now define a *mode vertex* of a connected graph G to be a vertex whose eccentricity occurs at least as often in $\mathrm{ES}(G)$ as the eccentricity of any other vertex in $\mathrm{ES}(G)$. The *mode* of a graph G is the subgraph induced by the mode vertices of G. Perhaps there are certain networks which require each eccentricity to occur the same number of times as every other eccentricity. For this we have the following definition. A *mode graph* is a connected graph for which each vertex is a mode vertex. An $(e_1, e_2, \cdots e_k)$-*mode graph* is a mode graph with eccentricities $e_1, e_2, \cdots e_k$.

If the radius of a graph G is equal to its diameter, then G is said to be *self-centered*. It is important to note that mode graphs are a generalization of self-centered graphs.

Figure 2 is a (2, 3, 4)-mode graph. The graph in Figure 1 is not a mode graph, but the subgraph induced by the vertices v, w, x and y is the mode of the graph since v, w, x and y are mode vertices.

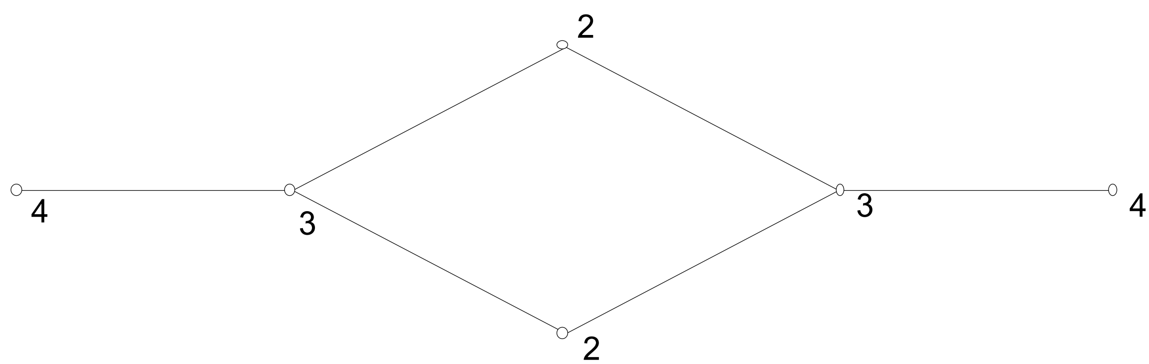

Figure 2: Mode graph.

CHAPTER 2

RESULTS

2.1 Induced Subgraphs of Mode Graphs

Any graph that is not a mode graph can be made into a mode graph by adding vertices and edges, or by just adding edges. Of course you can always add enough edges to make a graph a complete graph and thus a mode graph. It is interesting to see that by adding vertices and edges, a graph can be made into a mode graph without changing the eccentricities of the vertices of the original graph.

Theorem 2.1 *Any graph G can appear as an induced subgraph of a mode graph M, with the set of eccentricities of G is the same as the set of eccentricities of M.*

Proof. Let G be a graph such that $\mathrm{ES}(G) = e_1^{k_1}, e_2^{k_2}, \cdots e_n^{k_n}$. If $k_1 = k_2 = \cdots = k_n$, we are done. On the other hand, if $k_i < k_j$ for some i, j, we add a new vertex v, joining v to a vertex u with eccentricity e_i and also joining v to all vertices adjacent to u. Notice that $\mathrm{e}(v) = e_i$, and the eccentricities of all other vertices remain the same. This process can be repeated a finite number of times until $k_i = k_j$. Thus G is an induced subgraph of a mode graph M with $\mathrm{ES}(M) = e_1^k, e_2^k, \cdots, e_n^k$. $\square$

Notice that in the above proof if $\mathrm{e}(u) > 1$ then v need only be joined to the vertices adjacent to u and not to u itself.

In Figure 3, G is an induced subgraph of the mode graph M. This graph is obtained in the manner described in the proof of Theorem 2.1.

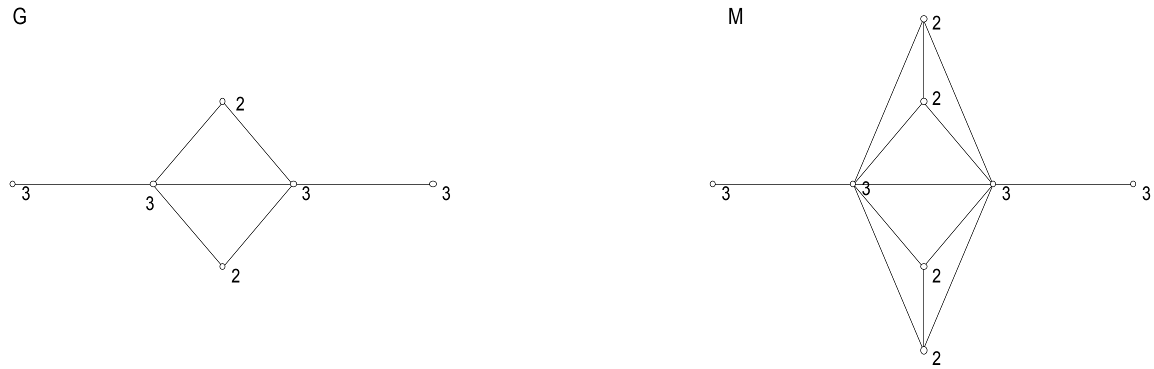

Figure 3: G is an induced subgraph of the mode graph M.

2.2 Cartesian Products

We can conceptualize the Cartesian product of two graphs G_1 and G_2, $G_1 \times G_2$ to be the graph obtained by placing a copy of G_2 at each vertex of G_1 and then joining corresponding vertices of G_2 for copies that are placed at adjacent vertices of G_1.

The *hypercube*, Q_n is defined recursively by $Q_1 = K_2$ and for $n \geq 2$, $Q_n = Q_{n-1} \times K_2$.

Theorem 2.2 *If G is a mode graph with eccentricities $j, j+1, \cdots, j+k$, then $G \times Q_i$ is a mode graph with eccentricities $j+i, j+1+i, \cdots, j+k+i$.*

Proof. Suppose G is a mode graph with vertex set $V(G) = v_1, v_2, \cdots, v_n$. Let G' be a copy of G with vertex set $V(G') = v'_1, v'_2, \cdots, v'_n$. Then $G \times Q_1$ is isomorphic to the graph $G \cup G'$ with the additional edges $v_p v'_p$ where $p = 1, 2, \cdots, n$. We can choose vertices $v_r, v_s \in V(G)$ such that $\mathrm{e}_G(v_r) = dist_G(v_r, v_s)$. Thus $\mathrm{e}_{G \times Q_1}(v_r) = dist_{G \times Q_1}(v_r, v'_s) = dist_G(v_r, v_s) + 1$. Also, $\mathrm{e}_{G \times Q_1}(v'_r) = dist_{G \times Q_1}(v'_r, v_s) = dist_G(v_r, v_s) + 1$. Thus for every vertex $v_r \in V(G)$ with $\mathrm{e}_G(v_r) = m$, there are exactly two vertices v_r and $v'_r \in V(G \times Q_1)$ with $\mathrm{e}_{G \times Q_1}(v_r) = \mathrm{e}_{G \times Q_1}(v'_r) = m + 1$. Thus $G \times Q_1$ is a mode graph. Notice that $Q_i = \overbrace{Q_1 \times Q_1 \cdots \times Q_1}^{i \text{ times}}$, and the Cartesian product of graphs is associative. Therefore $G \times Q_i$ is a mode graph since $G \times \overbrace{Q_1 \times Q_1 \cdots \times Q_1}^{i \text{ times}}$ is a mode graph. Furthermore, for every vertex with eccentricity m on $V(G)$ there are 2^i vertices with eccentricity $m + i$ in $V(G \times Q_i)$. $\square$

Let us now look at the more general case of graphs that are the Cartesian product of any two graphs. It is interesting to see that the eccentricity of a vertex in the Cartesian product of two graphs G_1 and G_2 can be easily calculated from the eccentricities of vertices in G_1 and G_2.

Theorem 2.3 *If* $\mathrm{e}_G(u) = a$ *and* $\mathrm{e}_H(v) = b$, *then the eccentricity of the vertex in* $G \times H$ *corresponding to* u *in* $V(G)$ *and* v *in* $V(H)$ *is* $a + b$.

Proof. Suppose $V(G)$ is $u_1, u_2, \cdots, u_n$, and $V(H)$ is $v_1, v_2, \cdots, v_m$. Then by placing a copy of H at each vertex of G and joining corresponding vertices of H that are placed at adjacent vertices of G we can define $V(G \times H)$ as $w_{1,1}, w_{1,2}, \cdots, w_{n,m}$ where $w_{i,j}$ is the vertex corresponding to the vertex v_j in H placed at the u_i copy of G. Since $G \times H$ only joins corresponding vertices of H in copies of H placed at adjacent vertices of G, the furthest vertex from $w_{i,j}$ is the vertex corresponding to the vertex v_k, where $\max_{v_p \in H} \mathrm{d}(v_p, v_j) = \mathrm{d}(v_k, v_j)$, placed at the u_r copy of G_1, where $\max_{u_t \in H} \mathrm{d}(u_t, u_i) = \mathrm{d}(u_r, u_i)$. This vertex is $w_{r,k}$. We can see that $\mathrm{d}_{G \times H}(w_{i,j}, w_{r,k}) = \mathrm{e}_{G \times H}(w_{i,j}) = \mathrm{e}_G(u_i) + \mathrm{e}_H(v_j)$ $\square$

Figure 4 illustrates this fact.

Notice that by Theorem 2.2, the hypercube, Q_1 is a self-centered graph with $\mathrm{rad}(Q_i) = i$. In fact Theorem 2.2 can be generalized to include the Cartesian product

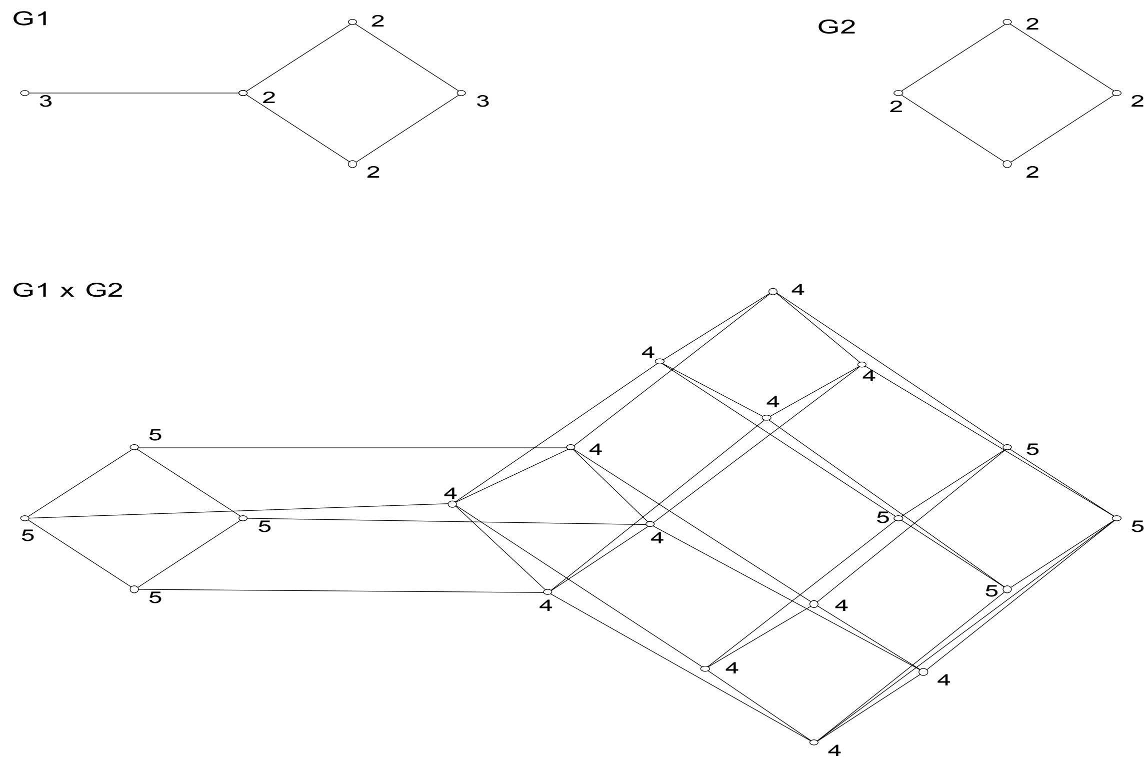

Figure 4: Eccentricities in the Cartesian product.

of a mode graph with any self-centered graph. This is a direct result of the proof of the following theorem which characterizes all mode graphs that are the Cartesian product of two graphs.

Theorem 2.4 *The Cartesian product of two graphs, $G_1 \times G_2$ is a mode graph if and only if G_1 is a mode graph and G_2 is self-centered.*

Proof. ($\Leftarrow$) If G_1 is a mode graph with $\mathrm{ES}(G_1) = e_1^k, e_2^k, \cdots, e_n^k$ and G_2 is a self centered graph with $\mathrm{ES}(G_2) = f^p$, then by Theorem 2.3 $\mathrm{ES}(G_1 \times G_2) = (e_1 + f)^{km}, (e_2 + f)^{km}, \cdots, (e_n + f)^{km}$. This implies that $G_1 \times G_2$ is a mode graph.

($\Rightarrow$) We will show this direction using the contrapositive. Case 1. Suppose G_1 is not a mode graph and G_2 is self-centered. Let $\mathrm{ES}(G_1) = e_1^{k_1}, e_2^{k_2}, \cdots, e_n^{k_n}, k_i \neq k_j$ for some i, j, and $\mathrm{ES}(G_2) = f^p$. Then, by Theorem 2.3, $\mathrm{ES}(G_1 \times G_2) = (e_1+f)^{k_1 \times p}, (e_2+f)^{k_2 \times p}, \cdots, (e_n+f)^{k_n \times p}$. Since $k_i p \neq k_j p$ for some i, j, $G_1 \times G_2$ is not a mode graph.

Case 2. Suppose neither G_1 nor G_2 is not self-centered. Let $\mathrm{ES}(G_1) = e_1^{k_1}, e_2^{k_2}, \cdots, e_n^{k_n}$ and $\mathrm{ES}(G_2) = f_1^{p_1}, f_2^{p_2}, \cdots, f_m^{p_m}$. By Theorem 2.3 there must be $k_1 p_1$ vertices with eccentricity $(e_1 + f_1)$ and $k_n p_m$ vertices with eccentricity $(e_n + f_m)$.

Subcase A. Without loss of generality suppose $m > n$. By Theorem 2.3 we also must have $k_1 p_m + k_2 p_{m-1} + \cdots + k_n p_{m-n+1}$ vertices with eccentricity $(e_1 + f_m)$ and $k_n p_1 + k_{n-1} p_2 + \cdots + k_1 p_n$ vertices with eccentricity $(e_n + f_1)$. For the sake of contradiction suppose $G_1 \times G_2$ is a mode graph. We have the following: $k_1 p_1 = k_n p_m$, $k_1 p_1 = k_1 p_m + k_2 p_{m-1} + \cdots + k_n p_{m-n+1}$ and $k_1 p_1 = k_n p_1 + k_{n-1} p_2 + \cdots + k_1 p_n$. Therefore $k_1 p_1 - k_1 p_m = k_2 p_{m-1} + \cdots + k_n p_{m-n+1}$ and $k_1 p_1 - k_n p_1 = k_{n-1} p_2 + \cdots + k_1 p_n$. Since G_1 and G_2 are not self-centered $k_1 p_1 - k_1 p_m > 0$ and $k_1 p_1 - k_n p_1 > 0$. Thus $p_1 > p_m$ and $k_1 > k_n$, which implies $k_1 p_1 > k_n p_m$. This is a contradiction.

Subcase B. Now suppose $m = n$. By Theorem 2.3 we also must have $k_1 p_m + k_2 p_{m-1} + \cdots + k_m p_1$ vertices with eccentricity $(e_1 + f_m) = (e_m + f_1)$. For the sake of contradiction suppose $G_1 \times G_2$ is a mode graph. We have the following: $k_1 p_1 = k_m p_m$ and $k_1 p_1 = k_1 p_m + k_2 p_{m-1} + \cdots + k_m p_1$. Therefore $k_1 p_1 - k_1 p_m = k_2 p_{m-1} + \cdots + k_m p_1$ and $k_1 p_1 - k_m p_1 = k_1 p_m + \cdots + k_{m-1} p_2$. Since G_1 and G_2 are not self-centered $k_1 p_1 - k_1 p_m > 0$ and $k_1 p_1 - k_m p_1 > 0$. Thus $p_1 > p_m$ and $k_1 > k_m$, which implies $k_1 p_1 > k_m p_m$. Again a contradiction.

Therefore $G_1 \times G_2$ is not a mode graph.

□

Corollary 2.5 *If G_1 is a mode graph with eccentricities $e, e+1, \cdots, e+k$, and G_2 is a* *self-centered graph with $rad(G_2) = i$ then $G_1 \times G_2$ is a mode graph with eccentricities $e+i, e+1+i, \cdots, e+k+i$.*

This is a direct result of Theorem 2.4

2.3 Eccentricity $(e_1, e_2, \cdots, e_n)$-Mode Graphs

Let $j, k \in N$. It has been shown in [1] that for any graph G, $\text{diam}(G) \leq 2\text{rad}(G)$, hence there is no graph G with eccentricities $k, k+1, \cdots, k+j$ where $j > k$. This is also the only restriction on possible eccentricities of a mode graph.

Theorem 2.6 *For any numbers $j, k \in N$, $j \leq k$, there exists a mode graph with eccentricities $k, k+1, \cdots, k+j$ where $j \leq k$.*

Proof. Let $i, j, k \in N$. The mode graph M_j in Figure 5 has vertices with eccentricities as shown. So M_j is a mode graph with eccentricities $j, j+1, \cdots, j+j$. By Theorem 2.8 $M_j \times Q_i$ is a mode graph with eccentricities $j+i, j+1+i, \cdots, j+j+i$. Let $k = i+j$. Hence there is a mode graph with eccentricities $k, k+1, \cdots, k+j$ where $j \leq k$. $\square$

An example of $M_j \times Q_i$ is shown in Figure 6.

Let G be a disconnected graph. We define $\text{rad}(G) = \infty$. The *join* $G = G_1 + G_2$ of two graphs has $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup \{uv | u \in V(G_1), v \in V(G_2)\}$.

Theorem 2.7 *A graph G is a (1, 2)-mode graph if and only if it is the join of a complete graph, G_1 and a second graph G_2, with $rad(G_2) \geq 2$ and $|V(G_1)| = |V(G_2)|$.*

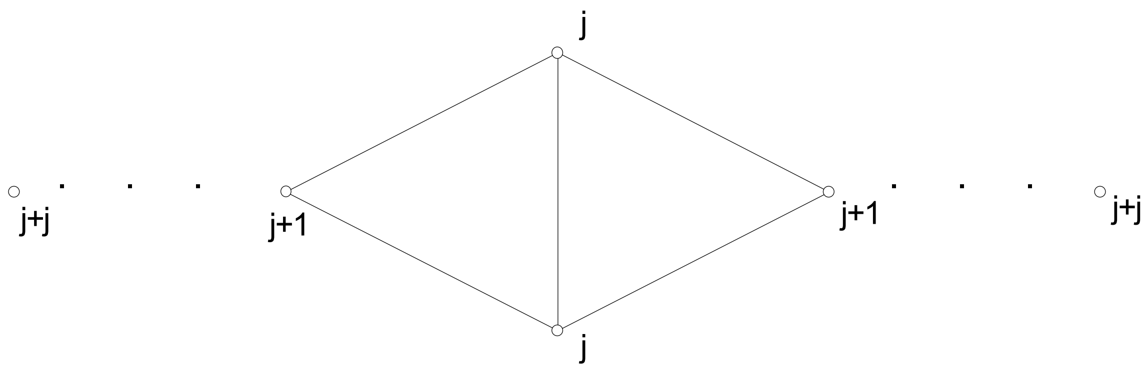

Figure 5: The graph M_j.

Proof. ($\Rightarrow$) Let G be a (1, 2)-mode graph. Now suppose G has n vertices of eccentricity 1. Then G must also have n vertices of eccentricity 2. Suppose G_1 is the induced subgraph of G on the n vertices of eccentricity one. Hence G_1 must be a complete graph on n vertices. Suppose G_2 is the induced subgraph of G on the n vertices of eccentricity 2. Let $v \in V(G_2)$. $\mathrm{e}(G_2(v) \geq 2$ since $\mathrm{e}_G(v) = 2$. Thus $\mathrm{rad}(G) \geq 2$. It is easy to verify that $G = G_1 + G_2$.
($\Leftarrow$) Let $G = G_1 + G_2$ where G_1 is a complete graph on n vertices, and G_2 is a graph on n vertices such that $\mathrm{rad}(G_2) \geq 2$. Thus for every vertex $v \in V(G_1), \mathrm{e}_{G_1}(v) = 1$ and for every vertex $u \in V(G_2), \mathrm{e}_{G_2}(u) \geq 2$. Hence, $\mathrm{e}_G(v) = 1$ and $\mathrm{e}_G(u) = 2$. Thus G is a (1, 2)-mode graph. $\square$

Figure 7 illustrates this theorem.

Since the vertices of the join of any two graphs must have eccentricity 1 or 2, if the join is a mode graph it must be either a (1, 2)-mode graph, a complete graph, or a self-centered graph with radius 2.

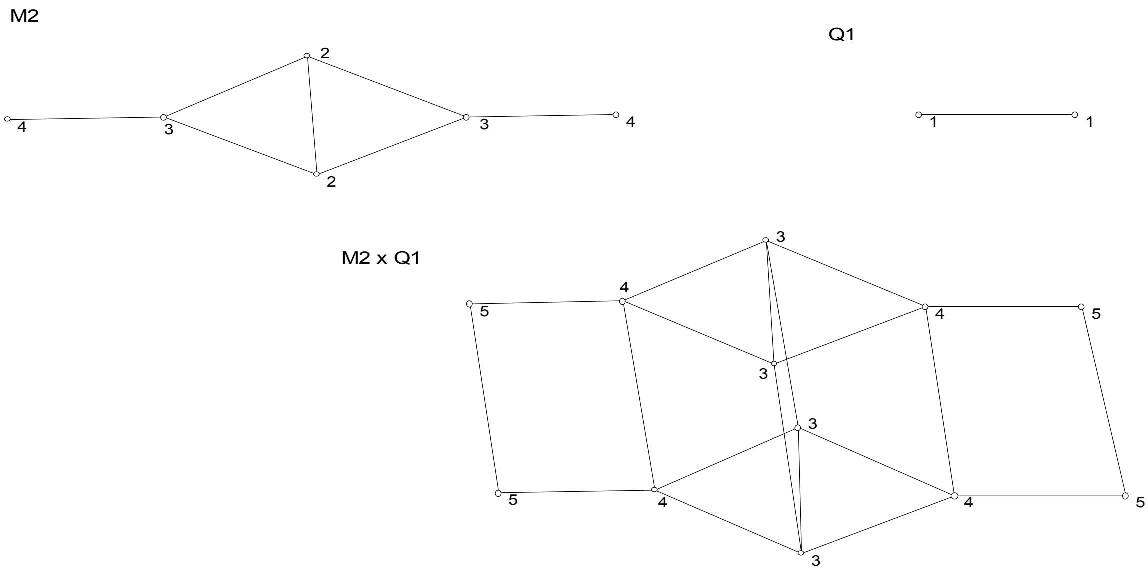

Figure 6: An example of M_j crossed with Q_i.

2.4 Mode Graphs Related to Various Other Graphs

We know that neither the Cartesian product of two mode graphs nor the join of two mode graphs are necessarily mode graphs. We next examine whether certain other graphs derived from mode graphs are necessarily mode graphs themselves.

Given a graph G, the *neighborhood graph*, $N(G)$, is the graph having the same vertices as G, but with two vertices adjacent if and only if there is a path in G of length two between them. The *distance 2 graph*, $D_2(G)$, is the graph having the same vertices as G, but with two vertices u and $v \in V(G)$ adjacent if and only if $\mathrm{d}_G(u, v) = 2$.

Theorem 2.8 *Given a mode graph G,* $\mathrm{N}G)$ *and* $\mathrm{D}_2(G)$ *are not necessarily mode graphs.*

Proof. Figure 8 illustrates this proof.

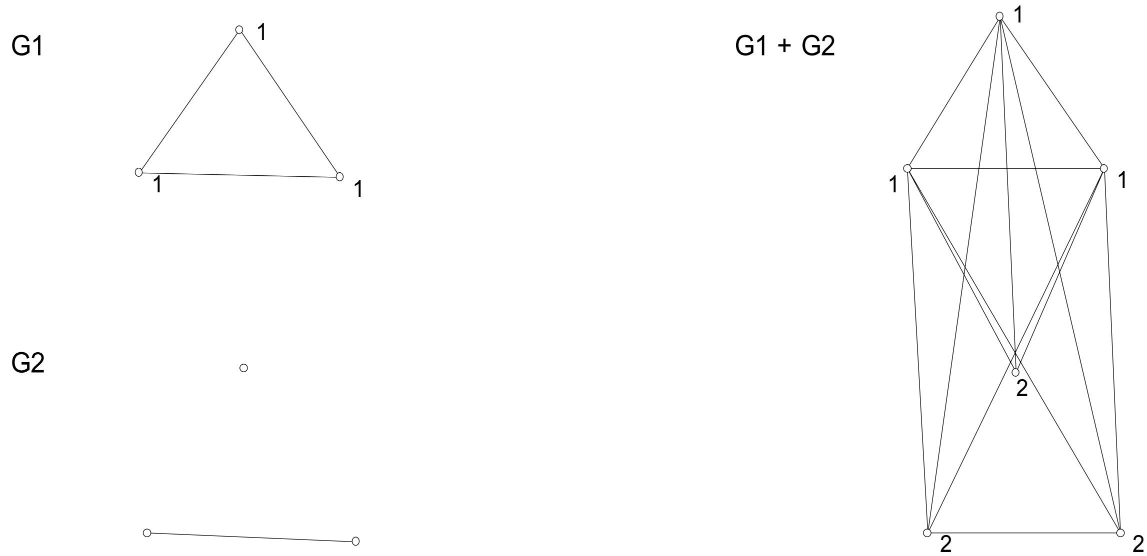

Figure 7: The join of G_1 and G_2 is a (1, 2)-mode graph.

As mentioned above, self-centered graphs are mode graphs. Therefore cycles, Q_i, and complete graphs are mode graphs. There are many mode graphs that are not self-centered, and there are many types of graphs that are mode graphs only under special circumstances.

Let $m = |V(\mathrm{Cen}(G))|$. If G is a mode graph, then $|V(G)| = km$, where $k = \mathrm{diam}(G) - \mathrm{rad}(G) + 1$. This is because there must be m vertices for each of $\mathrm{rad}(G), \mathrm{rad}(G) + 1, \cdots, \mathrm{diam}(G)$ eccentricities.

Theorem 2.9 *A tree T is a mode graph if and only if T is an even path.*

Proof. ($\Leftarrow$) Suppose T is an even path. Then T has eccentricity sequence $\mathrm{ES}(T) = (\frac{n}{2})^2, (\frac{n}{2} + 1)^2, \cdots, (n-1)^2$. Therefore T is a mode graph.

($\Rightarrow$) Suppose T is a mode graph, and for the sake of contradiction suppose T is not an even path. Any tree is either central or bicentral. Thus we have two cases.

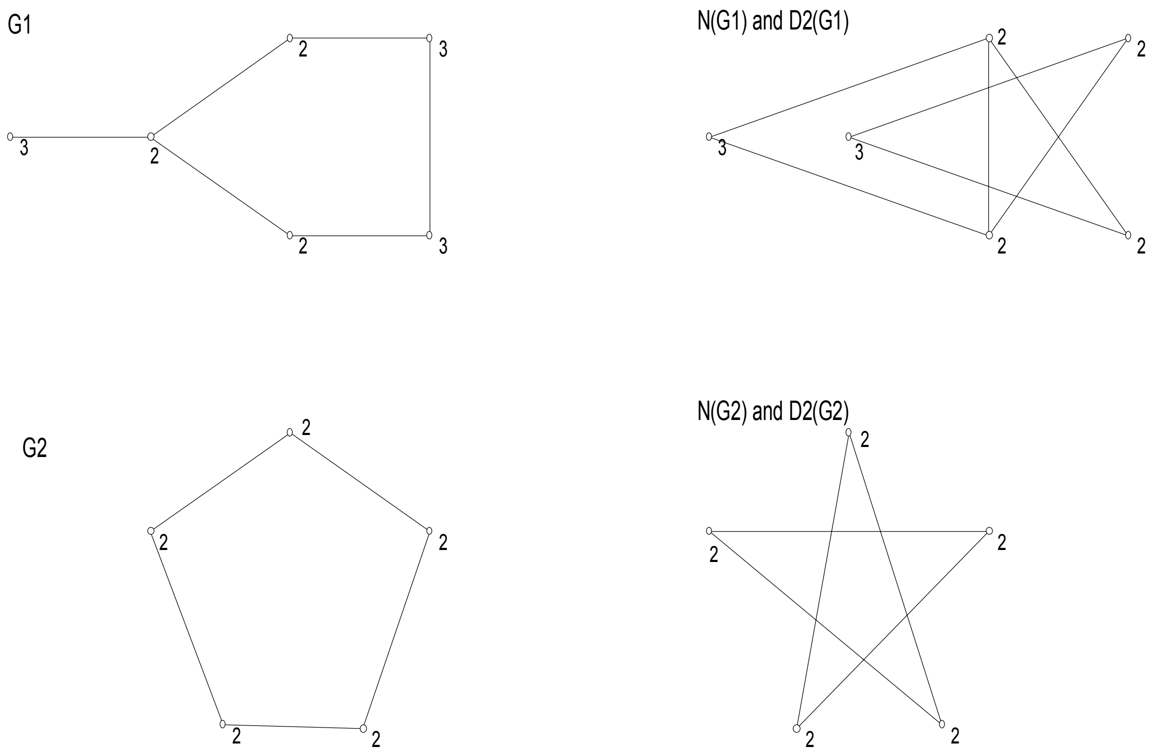

Figure 8: Neighborhood graphs and distance 2 graphs

Case 1: Suppose T is central. There is exactly one vertex with eccentricity $\mathrm{rad}(T)$. Thus $|V(T)| = \mathrm{diam}(T) - \mathrm{rad}(T) + 1$, since T is a mode graph. However, $|V(T)| \geq \mathrm{diam}(T) + 1$, since there are $\mathrm{diam}(T) + 1$ vertices in the longest path of T. Clearly, this is a contradiction.

Case 2: Suppose T is bicentral. There are exactly 2 vertices with eccentricity equal to $\mathrm{rad}(T)$. Also $\mathrm{rad}(T) = \frac{\mathrm{diam}(T)+1}{2}$ since the center of a tree is the same as the center of the longest path in the tree. Thus $|V(T)| = 2(\mathrm{diam}(T) - \mathrm{rad}(T) + 1) = 2\mathrm{diam}(T) - 2\mathrm{rad}(T) + 2 = 2\mathrm{diam}(T) - 2(\frac{\mathrm{diam}(T)+1}{2}) + 2 = \mathrm{diam}(T) + 1$. However, $|V(T)| > \mathrm{diam}(T) + 1$ since there are $\mathrm{diam}(T) + 1$ vertices in the longest path of T, and T is not itself a path. This is a contradiction. $\square$

We anticipated that, since even paths are the only trees that are mode graphs, it would be easy to characterize unicyclic graphs that are mode graphs. However, this characterization has not yet been determined.

If G is an n-cycle with vertices labeled $1, 2, \ldots, n$, $n \geq 4$, and α is a permutation of those vertices, then the graph $C(n, \alpha)$, called a *cycle permutation graph*, is the graph which consists of two copies of G, G_1 and G_2 along with the n edges obtained by joining i in G_1 with $\alpha(i)$ in G_2, $i = 1, 2, \ldots, n$.

Suppose $u, v \in V(C_n)$, and $u \neq v$. We have verified that, for $n \leq 7$, all of the permutations, α, that transpose u and v, for which $C(n, \alpha)$ is a mode graph, are those listed in Table 1.

Table 1: SINGLE TRANSPOSITIONS THAT ARE MODE GRAPHS

number of vertices in C_n	$d_{C_n}(u, v)$
4	1
4	2
5	1
5	2
6	1
6	3
7	1

Theorem 2.10 *Let $C(n, \alpha)$ be a cycle permutation graph where α is a single transposition. For $n \geq 7$, $C(n, \alpha)$ is a mode graph if and only if α transposes vertices which are adjacent in C_n, or if $n = 4m$, $m \in N$, α transposes vertices a distance $\frac{n}{2} - 1$ from each other in C_n.*

Proof. For $n < 11$ the theorem holds as previously verified. For simplicity let $n \geq$

11 and choose $k \leq \lfloor \frac{n}{2} + 1 \rfloor$. Let $C(n, \alpha)$ be a cycle permutation graph with vertex set $V = \{v_1, v_2, \cdots, v_n, u_1, u_2, \cdots, u_n\}$ and edge set $E = \{v_i v_{i+1}, u_i u_{i+1}, v_1 u_k, v_k u_1, v_i u_j\}, j \neq 1$ and $j \neq k$. Subscripts are read modulo n, and the symbol "n" is zero. We have the graph in Figure 9. Notice that this graph is isomorphic to the graph with the same vertex set but with edge set $E = \{v_i v_{i+1}, u_1 u_{i+1}, v_{1+l} u_{k+l}, v_{k+l} u_{1+l}, v_i u_j\}, j \neq 1 + l, k + l$.

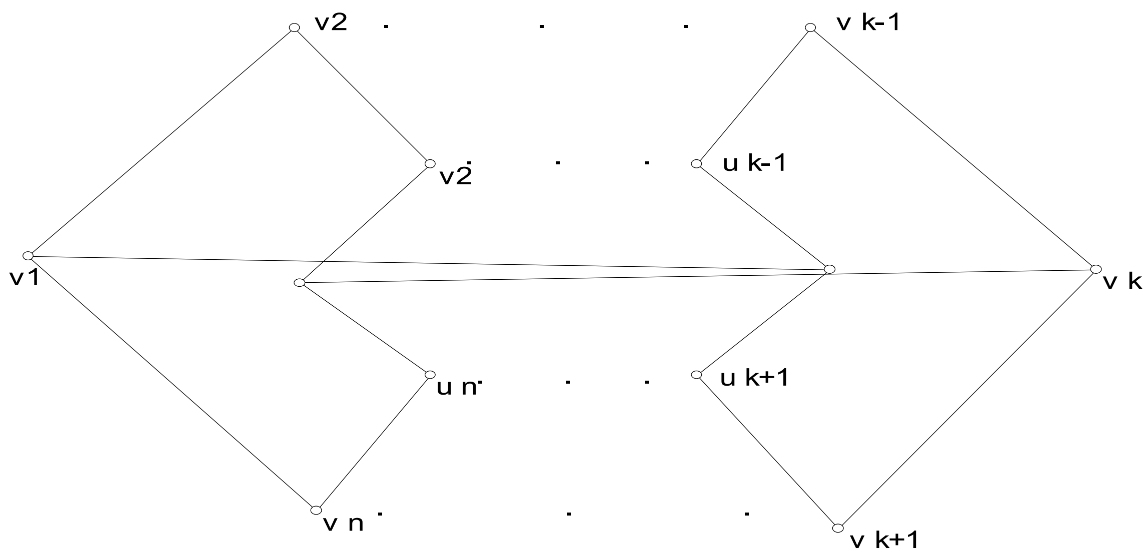

Figure 9: $C(n, \alpha)$ where α is a single permutation.

It can be seen because of symmetry that

$$\mathrm{e}(v_{\lfloor \frac{k-1}{2} + 1 \rfloor}) = \mathrm{e}(u_{\lfloor \frac{k-1}{2} + 1 \rfloor}) = \mathrm{e}(v_{\lceil \frac{k-1}{2} + 1 \rceil}) = \mathrm{e}(u_{\lceil \frac{k-1}{2} + 1 \rceil})$$

$$\mathrm{e}(v_{\lfloor \frac{k-1}{2} + 1 \rfloor} - 1) = \mathrm{e}(u_{\lfloor \frac{k-1}{2} + 1 \rfloor} - 1) = \mathrm{e}(v_{\lceil \frac{k-1}{2} + 1 \rceil} + 1) = \mathrm{e}(u_{\lceil \frac{k-1}{2} + 1 \rceil} + 1)$$

$$\cdots$$

$$\mathrm{e}(v_1) = \mathrm{e}(u_1) = \mathrm{e}(v_k) = \mathrm{e}(u_k)$$

$$\mathrm{e}(v_n) = \mathrm{e}(u_n) = \mathrm{e}(v_{k+1}) = \mathrm{e}(u_{k+1})$$

$$e(v_{n-1}) = e(u_{n-1}) = e(v_{k+2}) = e(u_{k+2})$$

$$\cdots$$

$$e(v_{\lceil\frac{n+1-k}{2}+k\rceil}) = e(u_{\lceil\frac{n+1-k}{2}+k\rceil}) = e(v_{\lfloor\frac{n+1-k}{2}+k\rfloor}) = e(u_{\lfloor\frac{n+1-k}{2}+k\rfloor})$$

Also note,

$$d(v_{\lfloor\frac{k-1}{2}+1\rfloor}, u_{\lfloor\frac{n+1-k}{2}+k\rfloor}) = \mathrm{diam}(C(n,\alpha)) = e(v_{\lfloor\frac{k-1}{2}+1\rfloor}).$$

Thus

$$d(v_{\lfloor\frac{k-1}{2}+1\rfloor}, u_{\lfloor\frac{n+1-k}{2}+k\rfloor}) = \begin{cases} \frac{n}{2}+1, & \text{if } k, n-k \text{ both odd} \\ \\ \lceil\frac{n}{2}\rceil, & \text{otherwise} \end{cases}.$$

Furthermore, for $n-k \geq 5$ which implies $n \geq 11$, $e(v_1) = \mathrm{rad}(C(n,\alpha)) = \lfloor\frac{n+1-k}{2}\rfloor + 1$.

($\Leftarrow$) Case 1: If $k=2$ then $\lfloor\frac{k-1}{2}+1\rfloor = 1$, so $e(v_1) = \mathrm{rad}(C(n,\alpha)) = \mathrm{diam}(C(n,\alpha))$. Hence $C(n,\alpha)$ is self-centered and therefore a mode graph.

Case 2: If $k=\frac{n}{2}$ and $n=4m, m \in N$. Then $\lfloor\frac{k-1}{2}+1\rfloor = \lfloor\frac{\frac{n}{2}-1}{2}+1\rfloor = \lfloor\frac{2m-1}{2}+1\rfloor = \lfloor m+\frac{1}{2}\rfloor = \lfloor\frac{n}{4}+\frac{1}{2}\rfloor = \frac{n}{4}$ and $\lceil\frac{k-1}{2}+1\rceil = \frac{n}{4}+1$. Also, $\lceil\frac{n+1-k}{2}+k\rceil = \lceil\frac{n+1-\frac{n}{2}}{2}+\frac{n}{2}\rceil = \lceil\frac{\frac{3}{2}n+1}{2}\rceil = \lceil\frac{3}{4}n+\frac{1}{2}\rceil = \frac{3}{4}n+1$ and $\lfloor\frac{n+1-k}{2}+k\rfloor = \frac{3}{4}n$.

Notice $e(v_n) = d(v_n, u_{\frac{1}{4}n+1}) = k+1-(\frac{1}{4}n+1)+1 = \frac{2n}{4}+\frac{4}{4}-\frac{n}{4}-\frac{4}{4} = \frac{n}{4}+1$ and $e(v_1) = \lfloor\frac{n}{2}+\frac{1}{2}-\frac{k}{2}\rfloor+1 = \lfloor\frac{n}{2}+\frac{1}{2}-\frac{n}{4}\rfloor+1 = \frac{n}{4}+1$. While $e(v_{\frac{n}{4}}) = \frac{n}{2}$. There is a path $v_1, v_2, \cdots, v_{\frac{n}{4}}$, and $e(v_1) = \frac{n}{4}+1, e(v_{\frac{n}{4}}) = \frac{n}{2}$. Since $\frac{n}{2}-(\frac{n}{4}+1) = \frac{2n}{4}-(\frac{n}{4}+1) = \frac{n}{4}-1$, every vertex in the v_1 to $v_{\frac{n}{4}}$ path has a different eccentricity. There is also a path $v_n, v_{n-1}, \cdots, v_{\frac{3n}{4}+1}$, and $e(v_n) = \frac{n}{4}+1, e(v_{\frac{3n}{4}+1}) = \frac{n}{2}$. Since $n-(\frac{3n}{4}+1) = \frac{n}{4}-1 = \frac{n}{2}-(\frac{n}{4}+1)$, every vertex in the v_n to $v_{\frac{3n}{4}+1}$ path has a different eccentricity. Combining the paths we have two vertices of each eccentricity, and each of the vertices is similar to 3 other vertices in the graph $C(n,\alpha)$. Therefore $C(n,\alpha)$ has 8 vertices each for eccentricities $\frac{n}{4}+1,\cdots,\frac{n}{2}$ and is thus a mode graph.

($\Rightarrow$). We will show this direction by the contrapositive. Suppose $k \neq 2$ and if $n = 4m, m \in N$ then $k \neq \frac{n}{2}$. Since $k \neq 2$, $\mathrm{e}(v_1) < \mathrm{e}(v_{\lfloor \frac{k-1}{2}+1 \rfloor}) = \mathrm{diam}(C(n,\alpha))$.

Case 1: Suppose n is odd. There must be 6 vertices in the periphery of $C(n,\alpha)$. There are at least 4 vertices with eccentricity less than $\mathrm{diam}(C(n,\alpha))$. Namely, v_1, u_1, v_k, u_k. If $\mathrm{e}(v_2) = \mathrm{e}(v_1)$ or $\mathrm{e}(v_n) = \mathrm{e}(v_1)$ then there are more than six vertices with eccentricity equal to $\mathrm{e}(v_1)$. If $\mathrm{e}(v_2) \neq \mathrm{e}(v_1)$ and $\mathrm{e}(v_n) \neq \mathrm{e}(v_1)$ then there are more than six vertices with eccentricity equal to $\mathrm{e}(v_1)$. Hence, $C(n,\alpha)$ is not a mode graph.

Case 2: Suppose $n = 4m-2$. Then if k is odd, $Per(C(n,\alpha))$ contains four vertices. However, there must be at least eight vertices with eccentricity equal to $\mathrm{diam}(C(n,\alpha)) - 1$. If k is even, there are eight vertices with eccentricity equal to $\mathrm{diam}(C(n,\alpha))$. That leaves $2n-8 = 8m-2-8$ vertices, and $8m-10$ is not divisible by 8, so $C(n,\alpha)$ cannot be a mode graph.

Case 3: Suppose $n = 4m$, $m \in N$. Thus $k \leq \frac{n}{2} - 1$. If k is odd, $Per(C(n,\alpha))$ contains four vertices. However, there must be at least 8 vertices with eccentricity equal to $\mathrm{diam}(C(n,\alpha)) - 1$. If k is even there are eight vertices with eccentricity equal to $\mathrm{diam}(C(n,\alpha)$. That leaves $2n-8$ vertices. Since $\mathrm{e}(v_1) = \lfloor \frac{n+1-k}{2} \rfloor + 1 \geq \lfloor \frac{n+1-(\frac{n}{2}-1)}{2} \rfloor + 1 = \lfloor \frac{n}{2} + \frac{1}{2} - \frac{n}{4} + \frac{1}{2} \rfloor + 1 = \frac{n}{4} + 2$, and $\mathrm{e}(v_n) = 2$, then there are at most $\frac{n}{2} - (\frac{n}{4}+2) + 1 = \frac{n}{4} - 1$ different eccentricities. That leaves $\frac{n}{4} - 2$ eccentricities for the remaining $2n-8$ vertices. However, $\frac{2n-8}{\frac{n}{4}-2} = \frac{8m-8}{m-2} = \frac{8(m-1)}{m-2} > 8$. Thus some eccentricities will occur more than 8 times. Hence $C(n,\alpha)$ is not a mode graph.

The *corona* of a graph H, $cor(H)$, is the graph obtained from H by adding a pendant edge to each vertex of H. We define the *k-corona* of a graph H, $kcor(H)$, to be the graph obtained by joining each vertex of H to an end-vertex of a path on k vertices. Figure 10 illustrates this concept.

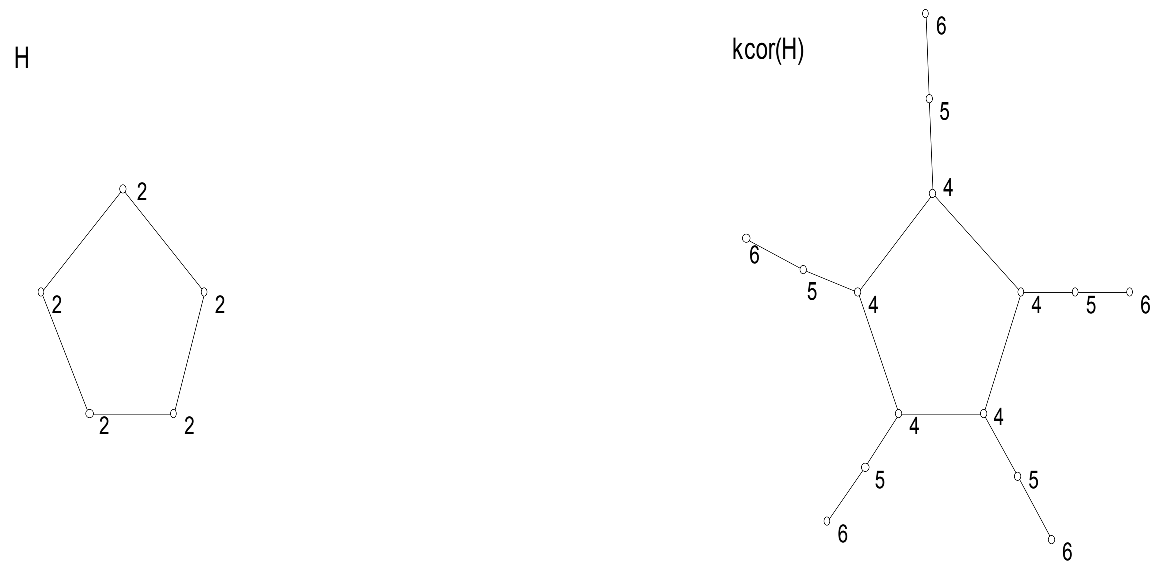

Figure 10: Example of k-corona

Theorem 2.11 *The k-corona of a graph H is a mode graph if and only if H is self-centered.*

Proof. ($\Leftarrow$). Suppose H is a self-centered graph with vertex set $V(H) = \{u_1, u_2, \cdots, u_n\}$. Then $\text{ES}(H) = e^n$. It can be seen that $\text{ES}(kcor(H)) = (e+k)^n, (e+k+1)^n, \cdots, (e+2k)^n$. Therefore $kcor(H)$ is a mode graph.

($\Rightarrow$) We show this direction by the contrapositive. Suppose H is not a self-centered graph and $V(H) = \{u_1, u_2, \cdots, u_n\}$. Then $\text{ES}(H) = e_1^{p_1}, e_2^{p_2}, e_m^{p_m}$ where $p_i \neq p_j$ for some i, j. We have $\text{ES}(kcor(H)) = (e_1 + k)^{p_1}, (e_2 + k)^{p_1+p_2}, \cdots$. Since $p_1 < p_1 + p_2$, $kcor(H)$ is not a mode graph.

$\square$

2.5 Mode Graphs With Given Size Periphery

Of course there are mode graphs that have any given size periphery greater than 1. This is because graphs such as complete graphs are mode graphs, and the periphery of these graphs can be any size greater than 1.

Theorem 2.12 *A graph G is a mode graph with $|periphery(G)| = 2$ if and only if G is an even path, or G is one of the graphs in figure 11.*

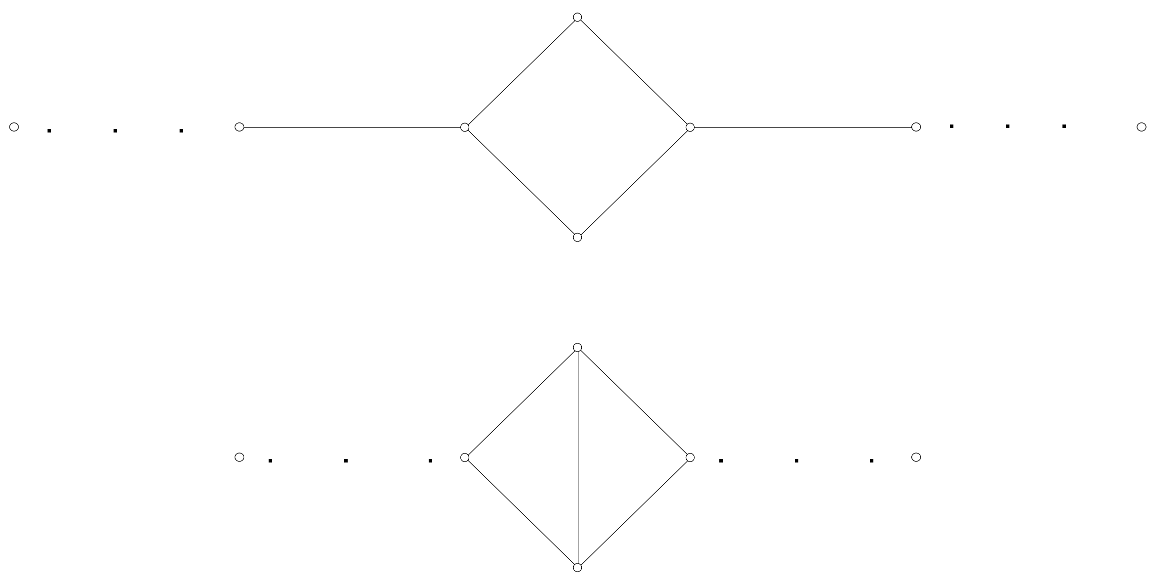

Figure 11: Other graphs G with $|periphery(G)| = 2$

Proof. ($\Leftarrow$) It is easily verified that even paths and the graphs in Figure 11 are mode graphs with $|periphery(G)| = 2$.

($\Rightarrow$) Suppose G is a mode graph, and $|periphery(G)| = 2$. We know $V(G)$ must contain an even number of vertices. In fact $|V(G)| = 2(\text{diam}(G) - \text{rad}(G) + 1)$. Since $\text{diam}(G) \leq 2\text{rad}(G)$, $|V(G)| \leq 2(\text{diam}(G) - \frac{\text{diam}(G)}{2} + 1) = \text{diam}(G) + 2$. We know there exists at least one path of length $\text{diam}(G)$ in G.

Case 1: Suppose this path is an even path. We know it must contain $\text{diam}(G)+1$ vertices. However, G has at most $\text{diam}(G)+2$ vertices, so $|V(G)| = \text{diam}(G)+1$ since $|V(G)|$ is even. This implies G is that even path.

Case 2: Suppose this path is an odd path, P_{2n-1}. This implies that $|V(G)| = \text{diam}(G)+2$, and there is one vertex, u in $V(G)$ that is not in $V(P_{2n-1})$. Since G is connected there is at least one edge joining P_{2n-1} and u. A single edge cannot join u to either of the end vertices of P_{2n-1} since the length of P_{2n-1} is equal to $\text{diam}(G)$. Thus by Theorem 2.9 at least two edges must join u to P_{2n-1}. If more than three edges join u to P_{2n-1} then the length of P_{2n-1} is greater than $\text{diam}(G)$. Also if u is joined to two vertices of P_{2n-1} that are more than a distance two apart, then the length of P_{2n-1} is greater than $\text{diam}(G)$. The center of P_{2n-1} is the lone vertex of P_{2n-1} with eccentricity equal to $\text{rad}(P_{2n-1})$. If u is joined by two edges to two adjacent vertices then u will have the same eccentricity as the higher of the eccentricities of the vertices it was joined to. This would create three vertices with the same eccentricity. Now it can be seen that two edges must join u to the vertices in P_{2n-1} that are adjacent to $\text{Cen}(P_{2n-1})$ in order for G to be a mode graph with two of each eccentricity. If there is a third edge it must join u to $\text{Cen}(P_{2n-1})$. This implies G is one of the graphs in Figure 11. $\square$

Given a sequence of consecutive natural numbers $e_1, e_2, \cdots, e_n$ we have found that mode graphs with these eccentricities, just like all graphs, are only restricted to $e_n \leq 2e_1$ since the diameter of a graph is no more than twice the radius. We have also found that the only restriction for the size of the periphery of a graph, call it k, is $k \geq 2$ since modes involve distance. However, when looking at the eccentricity sequence, $\mathrm{ES}(G) = e_1^k, e_2^k, \cdots, e_n^k$, of a mode graph G, which combines a list of consecutive eccentricities of a mode graph with the number of vertices in the periphery of the mode graph, we find there are further restrictions. These restrictions include but are not limited to $nk \geq e_n + k - 1$. This comes from the fact that a mode graph has the number of different eccentricities times the number of vertices in the periphery total vertices, and at least diameter plus one plus number of vertices in the periphery minus two total vertices. For instance there is not a mode graph G with $\mathrm{ES}(G) = \{3^2, 4^2\}$.

We know that if a graph exists with certain eccentricities, then a mode graph exists with those same eccentricities. One question which remains to be answered is, if $\mathrm{ES}(G) = e_1^{k_1}, e_2^{k_2}, \cdots, e_n^{k_n}$, under what conditions is there a mode graph M with $\mathrm{ES}(M) = e_1^k, e_2^k, \cdots, e_n^k$ such that $k < k_i$ for some i where $1 \leq i \leq n$.

BIBLIOGRAPHY

[1] G. Chartrand, L. Lesniak, *Graphs and Digraphs, Third Edition*, Chapman and Hall (1996).

[2] P. Torbett, Integer Sequences in Graphs, East Tennessee State University, Master's thesis, 1996.

[3] F. Buckley, F. Harary, *Distances in Graphs*, Redwood City, CA., Addison-Wesley (1990).

[4] J. Boland, R. Brigham, R. Dutton, Embedding Arbitrary Graphs in Neighborhood Graphs, *Journal of Combinatorics, Information, and System Sciences* 12 (1987) 101–112.

[5] R. Ringeisen, On Cycle Permutation Graphs, *Discrete Mathematics* 51 (1984) 265–275.

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