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VERTICES IN  
TOTAL DOMINATING SETS

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A Thesis

Presented to the Faculty of the Department of Mathematics

East Tennessee State University

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In Partial Fulfillment

of the Requirements for the Degree

Master of Science in Mathematical Sciences

---

by

Robert E. Dautermann III

May 2000

## APPROVAL

This is to certify that the Graduate Committee of

Robert Elmer Dautermann III

met on the

27th day of March, 2000.

The committee read and examined his thesis, supervised his defense of it in an oral examination, and decided to recommend that his study be submitted to the Graduate Council, in partial fulfillment of the requirements for the degree of Master of Science in Mathematics.

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## ABSTRACT

### VERTICES IN TOTAL DOMINATING SETS

by

Robert E. Dautermann III

Fricke, Haynes, Hedetniemi, Hedetniemi, and Laskar introduced the following concept [4]. For a graph  $G = (V, E)$ , let  $\rho$  denote a property of interest concerning sets of vertices. A vertex  $u$  is  $\rho$ -good if  $u$  is contained in a {minimum,maximum}  $\rho$ -set in  $G$  and  $\rho$ -bad if  $u$  is not contained in a  $\rho$ -set. Let  $g$  denote the number of  $\rho$ -good vertices and  $b$  denote the number of  $\rho$ -bad vertices. A graph  $G$  is called  $\rho$ -excellent if every vertex in  $V$  is  $\rho$ -good,  $\rho$ -commendable if  $g > b > 0$ ,  $\rho$ -fair if  $g = b$ , and  $\rho$ -poor if  $g < b$ . In this thesis the property of interest is total domination. The total domination number,  $\gamma_t$ , is the cardinality of a smallest total dominating set in a graph. We investigate  $\gamma_t$ -excellent,  $\gamma_t$ -commendable,  $\gamma_t$ -fair, and  $\gamma_t$ -poor graphs.

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## DEDICATION

To Gramma and Grampa Dautermann, whose collective dedication and perseverance permeated throughout the entire family.

## ACKNOWLEDGMENTS

I would like to thank the students and faculty of East Tennessee State University. Especially those who doubted me and those who encouraged me in this endeavor (sometimes one in the same). Of course, I need to thank the number 4. It has served me well for quite a long period of time; we would all be lost without it.

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# CHAPTER 1

## INTRODUCTION

A *graph*  $G = (V, E)$  is a finite nonempty set  $V(G)$  of objects called *vertices* together with a (possibly empty) set  $E(G)$  of unordered pairs of distinct vertices of  $G$  called *edges*. We will only consider simple graphs, those without directed edges or loops.

Let  $|V(G)| = n$  and  $|E(G)| = m$ . Two vertices of a graph are *adjacent* if there exists an edge between them. The *open neighborhood* of a vertex  $u$ , denoted as  $N(u)$ , consists of all vertices in  $V(G)$  which are adjacent to  $u$ . The *closed neighborhood* of a vertex  $v$ , is  $N[v] = N(v) \cup \{v\}$ . A graph in which every possible edge exists is called a *complete graph*, denoted  $K_n$ . The graph  $G_1$  in Figure 1 is a the complete graph on four vertices,  $K_4$ . For vertices  $u, v \in V(G)$ , a  *$u$ - $v$  path* is an alternating sequence of vertices and edges that begins with the vertex  $u$  and ends with the vertex  $v$  in which each edge of the sequence joins the vertex that precedes it in the sequence to the vertex that follows it in the sequence. Moreover, no vertex is repeated in this sequence. The number of edges in the sequence is considered the *length* of the path. A graph  $G$  is *connected* if for every pair of vertices in  $V(G)$ , there exists a path between them. A *cycle* on  $n$  vertices, denoted  $C_n$ , is a path which originates and concludes at the same vertex. The length of a cycle is the number of edges in the cycle. For example, the graph  $G_2$  in Figure 1 is a cycle of length 4,  $C_4$ . A *tree* is a connected graph which contains no cycles. An *endvertex* is any vertex of degree 1 (that is, a vertex adjacent to exactly one other vertex). A *support vertex* is any

vertex that is adjacent to at least one endvertex. A *bipartite graph*  $G$  is a graph with independent sets  $V_1$  and  $V_2$  where  $V_1$  and  $V_2$  partition  $V(G)$ . A *complete bipartite graph* is a bipartite graph with partite (disjoint) sets  $V_1$  and  $V_2$  having the added property that every vertex of  $V_1$  is adjacent to every vertex of  $V_2$ . Complete bipartite graphs are denoted  $K_{r,s}$ , where  $|V_1| = r$  and  $|V_2| = s$ .



Figure 1: Complete Graph  $K_4$  and Cycle  $C_4$

A set  $S$  is a *dominating set* of  $G$  if for each  $v \in V(G)$ ,  $v \in S$  or  $v$  is adjacent to a vertex in  $S$ . The *domination number*  $\gamma(G)$  is the minimum cardinality of a dominating set of  $G$ . A dominating set with cardinality  $\gamma(G)$  is called a  $\gamma$ -*set*. For a graph  $G$  with no isolated vertices, a *total dominating set*  $T$  is a set of vertices of  $G$  for which every  $v \in V(G)$  is adjacent to a vertex in  $T$ . The *total domination number*  $\gamma_t(G)$  is the minimum cardinality of any total dominating set of  $G$ . As before, a total dominating set with cardinality  $\gamma_t(G)$  is called a  $\gamma_t$ -*set*. In this thesis we are concerned with vertices in total dominating sets.

For an application of total domination, consider a mathematics conference where the attendees must form a committee to schedule the presentations. It would be

optimal to have a free flow of communication between the conference attendees and the committee and also among the committee members themselves. Therefore we want the committee to possess two desirable properties. First, that every non member know at least one member of the committee, for ease of communication. Second, each member of the committee should have an acquaintance on the committee, to avoid feelings of isolation and thus enhance co-operation [3]. For example, let Bill, Ted, Sara, and Marcia be four conference attendees. Suppose Bill knows only Ted and Marcia knows only Sara, but Ted and Sara know each other. Then both Ted and Sara must be on the committee, while Bill and Marcia can not be. Had Bill and Marcia been selected for the committee, then the second property would not be met and there would be a communication gap on the committee due to the isolation of both Bill and Marcia.

Consider a graph model of our conference where each person is represented by a vertex and two vertices are adjacent if the people represented by the vertices know each other. A committee with these properties is a total dominating set of the acquaintance graph of the conference attendees. If this is the smallest such committee, then we have a  $\gamma_t$ -set for the graph representing our conference. If we loosen the requirements and ask only for a committee comprised of individuals who collectively know every person at the conference, but not necessarily another committee member, then we have a dominating set. If this was the smallest such committee, then we would have a  $\gamma$ -set. In this thesis we investigate the total dominating sets of various graphs, based on the number of vertices of a graph which are contained in total dominating sets.

Fricke, Haynes, Hedetniemi, Hedetniemi, and Laskar [4] introduced the following concept. For a graph  $G = (V, E)$ , let  $\rho$  denote a property of interest of sets of vertices. We say that a vertex  $u$  is  $\rho$ -good if  $u$  is contained in a {minimum,maximum}  $\rho$ -set in  $G$  and  $\rho$ -bad if  $u$  is not contained in a  $\rho$ -set. Let  $g$  denote the number of  $\rho$ -good vertices and  $b$  denote the number of  $\rho$ -bad vertices. A graph  $G$  is called  $\rho$ -excellent if every vertex in  $V$  is  $\rho$ -good,  $\rho$ -commendable if  $g > b > 0$ ,  $\rho$ -fair if  $g = b$ , and  $\rho$ -poor if  $g < b$ . The property investigated in [4] was that of dominating sets. In other words, a vertex is  $\gamma$ -good if it is contained in some  $\gamma$ -set and a vertex is  $\gamma$ -bad if it is contained in no  $\gamma$ -set. A graph  $G$  is  $\gamma$ -excellent if every vertex in  $V(G)$  is  $\gamma$ -good,  $\gamma$ -commendable if  $g > b > 0$ ,  $\gamma$ -fair if  $g = b$ , and  $\gamma$ -poor if  $g < b$ .



Figure 2:  $P_4$  and  $P_4 \circ K_1$  Graphs

Before introducing our problem, we illustrate this concept with some examples. Since every graph has a dominating set, obviously every vertex-transitive graph is  $\gamma$ -excellent. In particular, cycles and complete graphs are  $\gamma$ -excellent. The 1-corona  $G \circ K_1$  associates with every vertex  $v_k \in V(G)$  a vertex  $u_k$  and joins the vertices  $v_k$  and  $u_k$  with the edge  $v_k u_k$ . In Figure 2, a path on four vertices,  $P_4$ , and the corona of the  $P_4$ ,  $P_4 \circ K_1$ , are represented. In this example, every vertex in  $V(P_4)$  now supports

an endvertex in  $P_4 \circ K_1$ . Consider the following two  $\gamma$ -sets. Let  $S$  be the set of all endvertices in  $V(P_4 \circ K_1)$ . Since each vertex in  $S$  is needed to dominate its support vertex,  $S$  is a  $\gamma$ -set. Let  $T$  be the set of all support vertices of  $V(P_4 \circ K_1)$ . Each vertex in  $T$  is needed to dominate its endvertex. Hence,  $T$  is a  $\gamma$ -set. Therefore, every vertex in  $V(P_4 \circ K_1)$  is  $\gamma$ -good and so  $P_4 \circ K_1$  is  $\gamma$ -excellent. In fact, using a similar argument, we can establish that all 1-coronas are  $\gamma$ -excellent [4].

For an example of  $\gamma$ -commendable graphs, consider a subdivided star  $K_{1,t}^*$  with  $t \geq 2$ . Since every endvertex or its support must be in every  $\gamma$ -set, every vertex in  $V(K_{1,t}^*)$  is  $\gamma$ -good except the center. Moreover, this center vertex will never be in any  $\gamma$ -set and for each endvertex there exists a  $\gamma$ -set containing it and another  $\gamma$ -set containing its support vertex. Thus,  $g = 2t$  and  $b = 1$ . Hence  $g > b$ , which implies that  $G$  is  $\gamma$ -commendable.

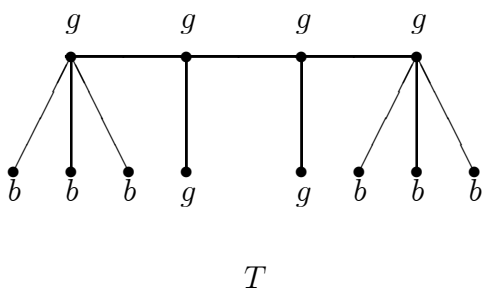


Figure 3: A  $\gamma$ -fair caterpillar.

For an example of a  $\gamma$ -fair graph, consider the caterpillar  $T$  in Figure 3. The good vertices are labeled  $g$  and the bad vertices are labeled  $b$ . For this graph,  $\gamma(T) = 4$  and  $T$  is clearly  $\gamma$ -fair [4].

Any star  $K_{1,t}$  for  $t \geq 2$  is  $\gamma$ -poor since the only good vertex in a star is the center.

Now we return to the problem of this thesis, where the desired property is total domination. In particular, we say that a vertex is  $\gamma_t$ -good if it is in some  $\gamma_t$ -set and  $\gamma_t$ -bad if it is in no  $\gamma_t$ -set. We investigate  $\gamma_t$ -excellent,  $\gamma_t$ -commendable,  $\gamma_t$ -fair, and  $\gamma_t$ -poor graphs. First in Chapter 2 we present some known results on total domination and then give an overview of the results from [4] on  $\gamma$ -excellent graphs. In Chapter 3 we illustrate our concept with examples and present some new results.

## CHAPTER 2

### LITERATURE SURVEY AND BACKGROUND

#### 2.1 Total Domination

Cockayne, Dawes, and Hedetniemi introduced the concept of total domination. Berge [1] presented the problem of the five queens, that is, how to place five queens on a chessboard so that every square is dominated by at least one queen [3]. It is easy to see that the solutions to this problem are dominating sets in the graph whose vertices represent the 64 squares of the chessboard and vertices  $a, b$  are adjacent if a queen may move from  $a$  to  $b$  in one move. Now extending the problem to include the property that not only must all squares be covered by a queen, but each queen must be covered by at least one other queen. This problem is that of total dominating sets, where all vertices are covered.

A total dominating set  $S$  is said to be *minimal* if when any vertex  $v \in S$  is removed from  $S$ , then  $S$  is no longer a total dominating set. The following theorem gives two properties pertaining to minimal total dominating sets.

**Theorem 2.1** [3] *If  $S$  is a minimal total dominating set of a connected graph  $G = (V, E)$ , then each  $v \in S$  has at least one of the following properties:*

$P_1$ : *There exists a vertex  $w \in V - S$  such that  $N(w) \cap S = \{v\}$ ;*

$P_2$ :  *$\langle S - \{v\} \rangle$  contains an isolated vertex.*

The following theorem gives an upper bound on the total domination number of a graph. Recall that  $n$  denotes the order of a graph, or the number of vertices in the



vertex set.

**Theorem 2.2** [3] *If  $G$  is a connected graph with  $n \geq 3$  vertices, then  $\gamma_t(G) \leq 2n/3$ .*

This theorem shows the best possible upper bound for  $\gamma_t(G)$ . Consider the path  $P_3$ .

For this path, it is obvious that  $\gamma_t(P_3)=2$ . Since  $(2 * 3)/3 = 2$ , the bound is sharp.

The following proof by Henning characterizes connected graphs of order at least 3 with total domination number exactly  $2/3$  their order. First let us define the  $k$ -corona. The  $k$ -corona of a graph  $G$  is the graph of order  $(k+1)|V(G)|$  obtained from  $G$  by identifying an endvertex  $v_j$  of a path of length  $k$  with each vertex  $v \in V(G)$  and attaching this path to  $v$  by letting  $v = v_j$ . The resulting paths are vertex disjoint.

**Theorem 2.3** [6] *Let  $G$  be a connected graph of order  $n \geq 3$ . Then  $\gamma_t(G) = 2n/3$  if and only if  $G$  is  $C_3$ ,  $C_6$ , or the 2-corona of some connected graph.*

A  $4/7$ -minimal graph  $G$  is edge-minimal with respect to satisfying the following three conditions:

- 1:  $\delta(G) \geq 2$
- 2:  $G$  is connected, and
- 3:  $\gamma_t(G) \leq 4n/7$ , where  $n$  is the order of  $G$ .

Henning further characterizes all  $4/7$ -minimal graphs by the following theorem.

**Theorem 2.4** [6] *If  $G$  is a connected graph of order  $n$  with minimum degree at least 2 and  $G \notin \{C_3, C_5, C_6, C_{10}\}$ , then  $\gamma_t(G) \leq 4n/7$ .*

For all paths  $P_n$ , the total domination numbers are known and easily verified. If  $n = 4k$ , then  $\gamma_t(P_n) = 2k$  and  $n \equiv 0(mod4)$ . If  $n = 4k + 1$ , then  $\gamma_t(P_n) = 2k + 1$  and

$n \equiv 1(\text{mod}4)$ . If  $n = 4k + 2$ , then  $\gamma_t(P_n) = 2k + 2$  and  $n \equiv 2(\text{mod}4)$ . If  $n = 4k + 3$ , then  $\gamma_t(P_n) = 2k + 2$  and  $n \equiv 3(\text{mod}4)$ . In Chapter 3 we will use these facts to characterize all  $\gamma_t$ -excellent,  $\gamma_t$ -commendable, and  $\gamma_t$ -fair paths.

## 2.2 $\gamma$ -excellent Graphs

From [4] we list several significant observations and results concerning  $\gamma$ -excellent graphs.

**Observation 2.5** [4] *For any connected graph  $G \neq K_2$ , there exists a  $\gamma$ -set containing all the support vertices of  $G$ .*

**Observation 2.6** [4] *For any  $\gamma$ -excellent graph  $G$ , every endvertex is in some  $\gamma$ -set and no endvertex is in every  $\gamma$ -set of  $G$ .*

For the next observation, consider a support vertex that is adjacent to two or more endvertices. In this case the support vertex must be in every  $\gamma$ -set. As a result, the endvertices will be in no  $\gamma$ -set. Hence, a graph with any support vertex adjacent to more than one endvertex is not  $\gamma$ -excellent.

**Observation 2.7** [4] *For any  $\gamma$ -excellent graph  $G$ , any support vertex is adjacent to exactly one endvertex.*

This observation can be seen more clearly if one considers a star,  $K_{1,t}$  for  $t \geq 2$ . The center vertex is adjacent to more than one endvertex. This center vertex dominates every vertex adjacent to it. It is easily shown that all stars are  $\gamma$ -poor.

**Proposition 2.8** [4] *Every graph is an induced subgraph of a  $\gamma$ -excellent graph.*

**Proof.** Consider any graph  $H$  and let  $G = H \circ K_1$ , the 1-corona of a graph  $H$ . Every vertex in  $V(H)$  is now a support vertex in  $G$ . Therefore,  $V(H)$  is a  $\gamma$ -set of  $G$ . As well, the set of endvertices in  $G$  is a  $\gamma$ -set. Hence every vertex in  $V(G)$  is in some  $\gamma$ -set and  $G$  is  $\gamma$ -excellent. Since  $H$  is an induced subgraph of  $G$ , every graph is an induced subgraph of some  $\gamma$ -excellent graph.  $\square$

The following proof characterizes all  $\gamma$ -excellent paths.

**Proposition 2.9** [4] *A path  $P_n$  is  $\gamma$ -excellent if and only if  $n = 2$  or  $n \equiv 1 \pmod{3}$ .*

**Proof.** It is a simple exercise to see that the paths  $P_2$  and  $P_n$  for  $n \equiv 1 \pmod{3}$  are  $\gamma$ -excellent. Let  $P_n$ ,  $n \geq 3$ , be a  $\gamma$ -excellent path and suppose that  $n \equiv 0, 2 \pmod{3}$ . If  $n \equiv 0 \pmod{3}$ , then  $P_n$  has a unique  $\gamma$ -set, which does not include all the vertices. If  $n \equiv 2 \pmod{3}$ , then no  $\gamma$ -set of  $P_n$  contains the third vertex on the path.  $\square$

The following theorem will show the connection between  $\gamma$ -excellent graphs and  $i$ -excellent, where  $i(G)$  is the *independent domination number*. The independent domination number is the minimum cardinality among all independent dominating sets of a graph  $G$ , where an independent dominating set is both independent and dominating.

**Theorem 2.10** [4] *If  $T$  is a  $\gamma$ -excellent tree, then  $\gamma(T) = i(T)$  and  $T$  is  $i$ -excellent.*

## CHAPTER 3

### EXAMPLES AND RESULTS

This chapter contains sections for each of the following  $\gamma_t$ -excellent,  $\gamma_t$ -commendable,  $\gamma_t$ -fair, and  $\gamma_t$ -poor graphs.

We begin with the following observation involving support vertices.

**Observation 3.11** *Every support vertex must be contained in every  $\gamma_t$ -set.*

**Observation 3.12** *An endvertex adjacent to two adjacent support vertices will never be contained in any  $\gamma_t$ -set.*

#### 3.1 $\gamma_t$ -excellent Graphs

Every complete graph, a graph containing all possible edges, is  $\gamma_t$ -excellent. Since all vertices are adjacent, the selection of any two vertices will form a  $\gamma_t$ -set. Since complete graphs are vertex transitive, every vertex is in some  $\gamma_t$ -set. Moreover, all vertex transitive graphs are  $\gamma_t$ -excellent. This includes all cycles,  $C_n$ , and all complete bipartite graphs,  $K_{r,r}$ . In fact, the complete bipartite graph  $K_{r,s}$  is  $\gamma_t$ -excellent for all  $r$  and  $s$ .

Our first proposition gives the  $\gamma_t$ -excellent paths. Label the vertices of the path  $P_n$  as  $v_1, v_2, \dots, v_n$ .

**Proposition 3.13** *Every path  $P_n$  for  $n = 3$  or  $n \equiv 2(\text{mod}4)$  is  $\gamma_t$ -excellent.*

**Proof.** Let  $P_n$  be a path for  $n = 3$  or  $n \equiv 2(\text{mod}4)$ . Obviously  $P_3$  is  $\gamma_t$ -excellent. For  $n \equiv 2(\text{mod}4)$ ,  $n = 4k + 2$  and  $\gamma_t(P_n) = 2k + 2$ . Note that  $P_n$  is  $\gamma_t$ -excellent for

$n = 2$  and  $n = 6$ . Assume some  $P_n$  is  $\gamma_t$ -excellent for some  $n = 4k + 2$ . To show  $P_n$  is  $\gamma_t$ -excellent for  $n = 4(k + 1) + 2$ , we must verify that each of the last four vertices of  $P_{4(k+1)+2}$  are in some  $\gamma_t$ -set. Since  $P_{4k+2}$  is assumed to be  $\gamma_t$ -excellent, then  $v_{4k+2} \in S$  for some  $\gamma_t$ -set  $S$ . Since  $v_{4k+2} \in S$ , then  $v_{4k+3}$  is dominated. This leaves  $v_{4k+4}, v_{4k+5}$ , and  $v_{4k+6}$  to totally dominate each other. Either  $v_{4k+4}$  and  $v_{4k+5}$  or  $v_{4k+5}$  and  $v_{4k+6}$  can be used to totally dominate these three vertices. So  $|S| + 2 = 2k + 2 + 2 = 2(k + 1) + 2$  and  $v_{4k+4}, v_{4k+5}$ , and  $v_{4k+6}$  are in some  $\gamma_t$ -set of  $P_{4(k+1)+2}$ . We need only to show that  $v_{4k+3}$  is in some  $\gamma_t$ -set with cardinality  $2(k + 1) + 2$ . To show this, consider a  $\gamma_t$ -set  $T$  for  $v_{4k+2}$  that contains  $v_{4k-1}$ . The set  $T$  exists since  $P_{4k+2}$  is  $\gamma_t$ -excellent. But  $v_{4k+2}$  is in  $T$  since it is a  $\gamma_t$ -set. So without loss of generality, let  $v_{4k+2} \in S$ . Then  $S - \{v_{4k+1}\} \cup \{v_{4k+3}, v_{4k+4}, v_{4k+5}\}$  is a  $\gamma_t$ -set for  $v_{4k+6}$  and has cardinality  $|S| - 1 + 3 = |S| + 2 = 2k + 2 + 2 = 2(k + 1) + 2$ . Hence,  $P_{4k+2}$  is  $\gamma_t$ -excellent.  $\square$

We now consider the induced subgraphs of  $\gamma_t$ -excellent graphs. In particular, we show that any graph  $G$  is an induced subgraph of some  $\gamma_t$ -excellent graph.

As defined in Chapter 2, the generalized 2-corona is obtained from a copy of a graph  $G$ , where for each vertex  $v \in V(G)$ , two new vertices  $v'$  and  $v''$ , and the edges  $vv'$  and  $v'v''$  are added. That is, for each vertex  $v \in V(G)$ , a pendant path of length 2 is added by identifying an endvertex of the new path  $P_3$  with  $v$ . Obviously,  $G$  is an induced subgraph of each, the 1-corona  $G \circ K_1$  and the 2-corona of  $G$ . Moreover,  $G$  is an induced subgraph of any  $k$ -corona of  $G$ .

**Proposition 3.14** *Every graph  $H$  is an induced subgraph of a  $\gamma_t$ -excellent graph.*

**Proof.** As we have seen, every graph  $H$  is an induced subgraph of the 2-corona of

$H$ . Let  $G$  be the 2-corona of a graph  $H$ . To see that  $G$  is  $\gamma_t$ -excellent, note that every  $\gamma_t$ -set of  $G$  must contain all the support vertices and a neighbor for each support vertex. If  $v$  is a support vertex in  $G$ , then  $v$  is in every  $\gamma_t$ -set of  $G$  and at least one neighbor of  $v$  is in every  $\gamma_t$ -set. Let  $S$  be the set of all support vertices in  $G$  and  $L$  be the set of all endvertices of  $G$ . Then  $S \cup L$  and  $S \cup V(H)$  are each  $\gamma_t$ -sets of  $G$ . Therefore  $G$  is  $\gamma_t$ -excellent. Since  $H$  is an induced subgraph of  $G$ , the proposition is true.  $\square$

**Corollary 3.15** *There does not exist a forbidden subgraph characterization of the class of  $\gamma_t$ -excellent graphs.*



Figure 4:  $P_4$  and 2-corona of  $P_4$  graphs.

The  $P_4$  and 2-corona of the  $P_4$  are in Figure 4. To help visualize the fore mentioned proof, one can consider these two graphs. In the 2-corona of the  $P_4$ , the four support vertices are in every  $\gamma_t$ -set. Let the set of support vertices be  $S$ , the set of endvertices be  $L$ , and the set of vertices in the  $P_4$  be  $V(H)$ . Clearly,  $S \cup L$  is a  $\gamma_t$ -set and  $S \cup V(H)$  is as well.

The graphs  $G_1$  and  $G_2$  in Figure 5 are each infinite families of  $\gamma_t$ -excellent graphs. In each graph, the support vertices are in every  $\gamma_t$ -set. As well, one neighbor of each

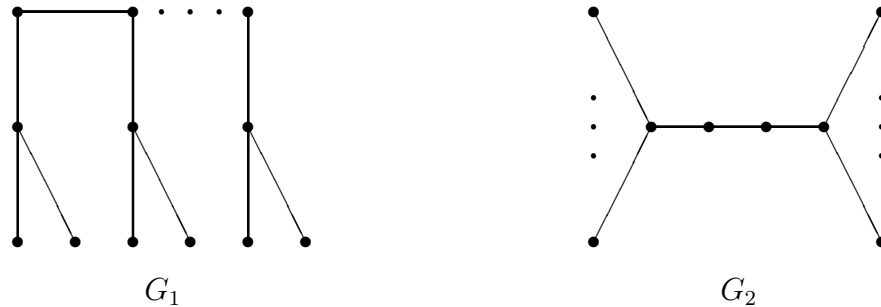


Figure 5: Infinite Families of  $\gamma_t$ -excellent Graphs.

support must be in every  $\gamma_t$ -set. It is a simple exercise to show that each of these graphs is  $\gamma_t$ -excellent.

### 3.2 $\gamma_t$ -commendable Graphs

Recall that a graph is  $\gamma_t$ -commendable if  $g > b > 0$ .

**Proposition 3.16** *Every 1-corona  $G = H \circ K_1$  with  $\Delta(H) \geq 1$  and  $\delta(H) = 0$  is  $\gamma_t$ -commendable.*

**Proof.** Let  $G = H \circ K_1$  with  $\Delta(H) \geq 1$  and  $\delta(H) = 0$ . This implies that the graph  $H$  has the connected subgraph(s)  $H_1, H_2, \dots, H_k$  and at least one isolate. For each  $H_n$ , every vertex in  $V(H_n)$  is now a support vertex in  $G$ . It follows that each  $V(H_n)$  is in a  $\gamma_t$ -set of  $G$ . Further, each isolate of  $H$  is a  $P_2$  in  $G$ . Obviously both vertices of a  $P_2$  are in every  $\gamma_t$ -set. Thus,  $g > n/2$  implies  $g > b$ . Now let  $u$ , an endvertex other than a vertex of a  $P_2$ , be in a  $\gamma_t$ -set  $S$ . Then the support vertex  $v$  of  $u$  must also be

in  $S$ , to totally dominate  $u$ . This contradicts the minimality of  $S$  as a  $\gamma_t$ -set. Hence not every vertex in  $G$  is in a  $\gamma_t$ -set, but  $g > b$ . Therefore  $G$  is  $\gamma_t$ -commendable.  $\square$

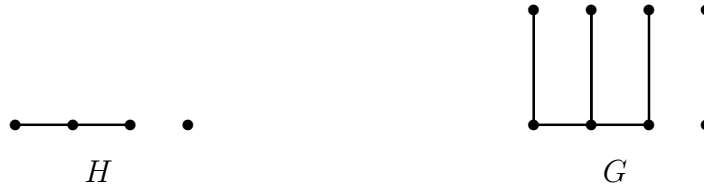


Figure 6:  $P_3 \cup K_1$  and  $(P_3 \cup K_1) \circ K_1$

For example, consider  $H = P_3 \cup K_1$  displayed in Figure 6. This graph is a  $P_3$  with a disjoint singleton vertex. The graph  $G = H \circ K_1$  in Figure 6 has  $\gamma_t(G) = 5$ . Each vertex in  $H$  is now a support vertex in  $G$ . Clearly each of these support vertices are needed to dominate their respective endvertices in  $G$ . Further, the  $K_1 \circ K_1$  has total domination number 2. Since each support vertex of  $G$  is in every  $\gamma_t$ -set, and both vertices of any  $P_2$  are in every  $\gamma_t$ -set, then  $G$  contains 5 good vertices and 3 bad vertices. Hence,  $G$  is  $\gamma_t$ -commendable. In general, for the 1-corona  $G = H \circ K_1$ ,  $\gamma_t(G) = |V(H)| + i$ , where  $i$  is the number of isolates in  $H$ , and  $G$  has a unique  $\gamma_t$ -set.

The path  $P_5$  is the path on five vertices. It is easily verifiable that  $\gamma_t(P_5) = 3$ . In fact,  $P_5$  has a unique  $\gamma_t$ -set. Since 3 of the 5 vertices of  $V(P_5)$  are in a  $\gamma_t$ -set, then  $g > b$  and  $P_5$  is  $\gamma_t$ -commendable. Notice that for  $P_5$ ,  $5 \equiv 1(\text{mod}4)$ . This leads us to our next proposition.

**Proposition 3.17** *Every path  $P_n$  for  $n \equiv 1(\text{mod}4)$  and  $n \equiv 3(\text{mod}4)$ , but  $n \neq 3$  is  $\gamma_t$ -commendable.*



**Proof.** Let  $P_n$  be a path and  $n \equiv 1(\text{mod}4)$ . Recall that for  $n \equiv 1(\text{mod}4)$ ,  $n = 4k + 1$  and  $\gamma_t(P_n) = 2k + 1$ . To be  $\gamma_t$ -commendable, there must be more good vertices than bad, but not every vertex can be good (this would imply  $\gamma_t$ -excellent). Clearly there are more good vertices than bad, since  $2k + 1 > (4k + 1)/2$ . For  $n = 5$ ,  $P_n$  has a unique  $\gamma_t$ -set and the endvertices are not included. Thus consider  $n \geq 9$ . Let the endvertex  $v_1 \in S$  for some  $\gamma_t$ -set  $S$ . Then  $v_2 \in S$  and the vertices  $v_1, v_2, v_3$  are totally dominated. Hence, there are  $n - 3$  vertices remaining to totally dominate. Since  $n = 4k + 1$ ,  $n - 3 = 4(k - 1) + 2$  and  $n - 3 \equiv 2(\text{mod}4)$ . It follows that  $\gamma_t(P_{n-3}) = 2(k - 1) + 2 = 2k$ . Therefore if an endvertex is in a  $\gamma_t$ -set  $S$ , then  $S$  has cardinality  $2k + 2$ , which is a contradiction since  $\gamma_t(P_{4k+1}) = 2k + 1$ . Hence, not every vertex is in a  $\gamma_t$ -set, but  $g > b$ . Therefore,  $P_n$  for  $n \equiv 1(\text{mod}4)$  is  $\gamma_t$ -commendable.

Next let  $P_n$  be a path and  $n \equiv 3(\text{mod}4)$  but  $n \neq 3$ . Clearly  $g > b$  for  $P_n$  since  $\gamma_t(P_n) = 2k + 2 > (4k + 3)/2$ . Since  $n \equiv 3(\text{mod}4)$ , then  $n = 4k + 3$ . Suppose  $v_4 \in S$ , the fourth vertex from either end of the path, for some  $\gamma_t$ -set  $S$ . Then there are subgraphs  $P_3$  and  $P_{4(k-1)+3}$  to totally dominate. But  $\gamma_t(P_3) = 2$  and  $\gamma_t(P_{4(k-1)+3}) = 2(k - 1) + 2 = 2k$ , implying that  $2k + 2 + 1$  vertices are needed to totally dominate a  $P_{4k+3}$  if  $v_4$  is in  $S$ . This is a contradiction since  $\gamma_t(P_{4k+3}) = 2k + 2 \neq 2k + 3$ . Hence, not every vertex is in a  $\gamma_t$ -set. Therefore,  $P_n$  for  $n \equiv 3(\text{mod}4)$  is  $\gamma_t$ -commendable.  $\square$

We now consider the induced subgraphs of  $\gamma_t$ -commendable graphs. In particular, we show that any graph  $H$  is an induced subgraph of some  $\gamma_t$ -commendable graph.

**Proposition 3.18** *Every graph  $H$  is an induced subgraph of a  $\gamma_t$ -commendable graph.*

**Proof.** Let  $H$  be a graph. Let  $G$  be the 3-corona of  $H$  with the following property. For each vertex  $v_i \in V(H)$ , add vertex  $v_i'$  and edge  $v_i v_i'$  to the 3-corona of  $H$ . Clearly

$H$  is an induced subgraph of  $G$ . Now we need only show that  $G$  is  $\gamma_t$ -commendable, or that  $g > b > 0$ .

Let  $H_1, H_2, \dots, H_k$  be components of  $H$ . Either  $|V(H_i)| > 1$  or  $|V(H_i)| = 1$ . Let  $|V(H_i)| = 1$ . This implies that  $H_i$  is an isolate, say the vertex  $u_i$ . In  $G$ ,  $u_i$  is a support vertex of a  $P_5$ . A  $P_5$  has a unique  $\gamma_t$ -set and, it follows that  $\gamma_t(P_5) = 3$  and every  $P_5$  is  $\gamma_t$ -commendable. Therefore, for every isolated vertex in  $H$ , we have a  $\gamma_t$ -commendable subgraph in  $G$ .

Now let  $|V(H_j)| > 1$ . This implies that each  $v_j \in V(H_j)$  is a support vertex in  $G$ . Moreover, each  $v_j$  is adjacent to at least 1 other support vertex,  $v_h \in G$ . These support vertices dominate each other, and by Observation 3.12 the endvertices  $v_j'$  and  $v_h'$  are not in any  $\gamma_t$ -set of  $G$ . Furthermore, these are the only bad vertices of  $G$ . Since each subgraph of  $G$  is  $\gamma_t$ -commendable, it follows that the  $g > b$  for the entire graph  $G$ . Hence,  $G$  is  $\gamma_t$ -commendable and the proposition is true.  $\square$

**Corollary 3.19** *There does not exist a forbidden subgraph characterization of the class of  $\gamma_t$ -commendable graphs.*

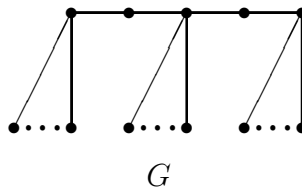


Figure 7: Infinite Family of  $\gamma_t$ -commendable Graphs.

The graph  $G$  in Figure 7 is another infinite family of  $\gamma_t$ -commendable graphs. Each support vertex in this graph is in every  $\gamma_t$ -set. However, the only endvertices in any  $\gamma_t$ -set are those which are adjacent to the outermost support vertices. The endvertices adjacent to the center support vertex are in no  $\gamma_t$ -set. If each support is adjacent to the same number of support vertices, the resultant graph is  $\gamma_t$ -commendable.

### 3.3 $\gamma_t$ -fair Graphs

For a graph to be  $\gamma_t$ -fair, we must have the same number of good vertices as bad. This leads us to the following observation.

**Observation 3.20** *All  $\gamma_t$ -fair graphs must have even order.*

Consider the path  $P_4$ . This graph has a unique  $\gamma_t$ -set (only the two central vertices are in a  $\gamma_t$ -set) and  $\gamma_t(P_4) = 2$ . Exactly half of the vertices in  $V(P_4)$  are good and half are bad. This implies that  $P_4$  is  $\gamma_t$ -fair.

We now have the following proposition concerning  $\gamma_t$ -fair paths.

**Proposition 3.21** *A path  $P_n$  is  $\gamma_t$ -fair if and only if  $n \equiv 0(\text{mod}4)$ .*

**Proof.** Let  $P_n$  be a path and  $n \equiv 0(\text{mod}4)$ . Then  $n = 4k$  and  $\gamma_t(P_n) = 2k$ . Since every  $P_{4k}$  has a unique  $\gamma_t$ -set,  $P_{4k}$  is  $\gamma_t$ -fair.

Assume  $P_n$  is  $\gamma_t$ -fair. Then from Proposition 3.13 and Proposition 3.17,  $n \neq 3$  and  $n \not\equiv 1, 2, 3(\text{mod}4)$ . Hence,  $n \equiv 0(\text{mod}4)$ .  $\square$

**Proposition 3.22** *Any connected graph  $G = H \circ K_1$  is  $\gamma_t$ -fair when  $|V(H)| \geq 2$ .*

**Proof.** Let  $G = H \circ K_1$  be a connected graph. Clearly  $|V(G)|=2|V(H)|$ , so the order of  $G$  is even. Since all vertices in the subgraph  $H$  of  $G$  are needed to dominate their endvertices and each vertex in  $H$  is adjacent to another vertex in  $H$ ,  $\gamma_t(G) = |V(H)|$ , where  $V(H)$  is a total dominating set of  $G$ . Notice that  $V(H)$  is the set of all support vertices of  $G$ . Since all support vertices must be contained in every  $\gamma_t$ -set, the inclusion of any endvertex would violate the minimality of a  $\gamma_t$ -set. Hence,  $G$  is  $\gamma_t$ -fair.  $\square$

**Corollary 3.23** *Every connected graph  $H$  is the induced subgraph of a  $\gamma_t$ -fair graph.*

From Corollary 3.23 we see that there is no induced subgraph characterization of  $\gamma_t$ -fair graphs.

The graph  $P_4 \circ K_1$  in Figure 2 is the 1-corona of the connected graph  $P_4$ . The order  $n$  of this graph is even, which is a necessary condition for  $\gamma_t$ -fair. Notice that the support vertices of this graph are the unique  $\gamma_t$ -set and no endvertex can be in any  $\gamma_t$ -set.



Figure 8:  $\gamma_t$ -fair Graphs.

Each of the graphs  $G_1$  and  $G_2$  in Figure 8 are  $\gamma_t$ -fair. For graph  $G_1$ ,  $\gamma_t(G_1) = 3$ . The only good vertices are each of the two support vertices and the two central vertices of degree 4. The remaining 4 vertices will be in no  $\gamma_t$ -set. The graph  $G_2$  is an infinite family of  $\gamma_t$ -fair graphs. Notice that exactly 1 of the support vertices is adjacent to exactly 2 endvertices. The center and the support vertices form the unique  $\gamma_t$ -set of  $G_2$ .

### 3.4 $\gamma_t$ -poor Graphs

A graph  $G$  is  $\gamma_t$ -poor if  $g < b$ . We have shown in the previous three sections that all the paths  $P_n$  for  $n \equiv 2(\text{mod}4)$  is  $\gamma_t$ -excellent,  $n \equiv 1(\text{mod}4)$  and  $n \equiv 3(\text{mod}4)$  are  $\gamma_t$ -commendable, and  $n \equiv 0(\text{mod}4)$  are  $\gamma_t$ -fair. Therefore we have the following corollary.

**Corollary 3.24** *There does not exist a  $\gamma_t$ -poor path  $P_n$ .*

We now consider the induced subgraphs of  $\gamma_t$ -poor graphs. In particular, we show that any graph  $H$  is an induced subgraph of some  $\gamma_t$ -poor graph.

Let  $H$  be a graph. Define  $G = H \circ I_j$ , where  $I_j$  is  $j$  isolates, as, for every vertex  $v \in V(H)$ , add  $j \geq 2$  endvertices adjacent to  $v$ . Clearly,  $H$  is an induced subgraph of  $G$ .

**Proposition 3.25** *Every graph  $H$ , where  $\delta(H) \geq 1$ , is an induced subgraph of a  $\gamma_t$ -poor graph.*

**Proof.** Let  $G = H \circ I_j$ , where  $\delta(H) \geq 1$ . As we have previously seen,  $H$  is an induced subgraph of  $G$ . To prove this proposition, we need only show that  $G$  is

$\gamma_t$ -poor. Since  $\delta(H) \geq 1$ , every vertex  $v \in V(H)$  has a neighbor in  $V(H)$ . Notice that each vertex  $v \in V(H)$  is a support vertex in  $G$ . Therefore, every support vertex in  $G$  is adjacent to another support vertex in  $G$ , dominating one another and their adjacent endvertices. This implies that  $G$  has a unique  $\gamma_t$ -set,  $V(H)$ . For each vertex  $v \in V(H)$ ,  $v$  has  $j$  support vertices adjacent to it, of which none will be in any  $\gamma_t$ -set. By definition,  $j \geq 2$ . This implies that for  $|V(H)|$  good vertices, there exist  $j|V(H)|$  bad vertices. Clearly there exist more bad vertices in  $G$  than good vertices. Hence,  $G$  is  $\gamma_t$ -poor and the proposition is true.  $\square$

**Corollary 3.26** *There does not exist a forbidden induced subgraph characterization of the class of  $\gamma_t$ -poor graphs.*



Figure 9: Infinite Families of  $\gamma_t$ -poor Graphs

In each of the  $\gamma_t$ -poor graphs in Figure 9, all of the support vertices must be in every  $\gamma_t$ -set. Each of these graphs has a unique  $\gamma_t$ -set, the support vertices. Hence  $b > g$ , which implies  $\gamma_t$ -poor.

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