5-2000

Performance of bootstrap confidence intervals for L-moments and ratios of L-moments.

Suzanne Glass
East Tennessee State University

PERFORMANCE OF BOOTSTRAP CONFIDENCE INTERVALS FOR L-MOMENTS AND RATIOS OF L-MOMENTS

A Thesis

Presented to the Faculty of the Department of Mathematics

East Tennessee State University

In Partial Fulfillment

of the Requirements for the Degree

Master of Science in Mathematical Sciences

---

by

Suzanne P. Glass

May 2000

APPROVAL

This is to certify that the Graduate Committee of

Suzanne P. Glass

met on the

27th day of March, 2000.

The committee read and examined her thesis, supervised her defense of it in an oral examination, and decided to recommend that her study be submitted to the Graduate Council, in partial fulfillment of the requirements for the degree of Master of Science in Mathematics.

Dr. Edith Seier
Chair, Graduate Committee

Dr. Price

Mr. Jablonski

Signed on behalf of
the Graduate Council

Dr. Wesley Brown
Dean,
School of Graduate Studies

ABSTRACT

PERFORMANCE OF BOOTSTRAP CONFIDENCE INTERVALS FOR

L-MOMENTS AND RATIOS OF L-MOMENTS

by

Suzanne P. Glass

L-moments are defined as linear combinations of expected values of order statistics of a variable.(Hosking 1990) L-moments are estimated from samples using functions of weighted means of order statistics. The advantages of L-moments over classical moments are: able to characterize a wider range of distributions; L-moments are more robust to the presence of outliers in the data when estimated from a sample; and L-moments are less subject to bias in estimation and approximate their asymptotic normal distribution more closely.

Hosking (1990) obtained an asymptotic result specifying the sample L-moments have a multivariate normal distribution as $n \to \infty$. The standard deviations of the estimators depend however on the distribution of the variable. So in order to be able to build confidence intervals we would need to know the distribution of the variable.

Bootstrapping is a resampling method that takes samples of size n with replacement from a sample of size n. The idea is to use the empirical distribution obtained with the subsamples as a substitute of the true distribution of the statistic, which we ignore. The most common application of bootstrapping is building confidence intervals without knowing the distribution of the statistic.

The research question dealt with in this work was: How well do bootstrapping confidence intervals behave in terms of coverage and average width for estimating L-moments and ratios of L-moments? Since Hosking's results about the normality of the estimators of L-moments are asymptotic, we are particularly interested in knowing how well bootstrap confidence intervals behave for small samples.

There are several ways of building confidence intervals using bootstrapping. The most simple are the standard and percentile confidence intervals. The standard confidence interval assumes normality for the statistic and only uses bootstrapping to estimate the standard error of the statistic. The percentile methods work with the $(\alpha/2)$th and $(1-\alpha/2)$th percentiles of the empirical sampling distribution. Compar-

ing the performance of the three methods was of interest in this work.

The research question was answered by doing simulations in Gauss. The true coverage of the nominal 95% confidence interval for the L-moments and ratios of L-moments were found by simulations.

DEDICATION

This thesis is dedicated to my husband, Jason, and my son, Jeffrey, who have patiently been by my side as I have worked on my graduate degree. Thanks for believing in me. I love you.

ACKNOWLEDGEMENTS

A special thanks to my GOD. Through my faith in Him all things are possible. A special thanks is also given to my thesis advisor, Dr. Edith Seier, who has been patient and a joy to work with the past year.

Contents

List of Tables

List of Figures

CHAPTER 1

L-MOMENTS

For a variable X with density function $f(x)$ and distribution function $F(x)$, the classical moments of order r with respect to an arbitrary point a is defined as

$$\mu_r' = \int_{-\infty}^{\infty} (x-a)^r dF.$$

Moments are used to characterize probability distributions. The first moment, with respect to the origin (a=0, r=1) is $E(X)$, the mean of the distribution, and is an indicator of location. The second moment, with respect to the mean ($a = \mu$, r=2), $E(X-\mu)^2$ is the variance and a measure of spread. The two measures of shape we are interested in, skewness and kurtosis, are ratios of moments. According to Groeneveld (1991), "positive skewness results from a location- and scale-free movement of the probability mass of a distribution. Mass at the right of the median is moved to from the center to the right tail of the distribution, and simultaneously mass at the left of the median is moved to from the center to the left of the distribution." The classical measure of skewness is

$$\sqrt{\beta_1} = \frac{E(X-\mu)^3}{(\sqrt{E(X-\mu)^2})^3} = \frac{\mu_3'}{(\sqrt{\mu_2'})^3}.$$

Kurtosis is defined as "the location- and scale-free movement of probability mass from the shoulders of a distribution into its center and tails . . . (which) can be formalized in many ways" [1]. The classical measure of kurtosis is

$$\beta_2 = \frac{E(X-\mu)^4}{[E(X-\mu)^2]^2} = \frac{\mu_4'}{(\sqrt{\mu_2'})^4}$$

In this chapter another type of moments, the L-moments and ratios of L-moments, defined by Hosking (1990) will be examined and compared with the classical moments.

1.1 Definitions of L-moments, L-skewness and L-kurtosis

L-moments are defined as linear combinations of expected values of order statistics of a variable. Given X a random variable with density function f and $E(X) < \infty$. The L-moments are defined as:

$$L_1 = E(X_{1:1})$$

$$L_2 = \frac{1}{2}E(X_{2:2} - X_{1:2})$$
$$L_3 = \frac{1}{3}E(X_{3:3} - 2X_{2:3} + X_{1:3})$$
$$L_4 = \frac{1}{4}E(X_{4:4} - 3X_{3:4} + 3X_{2:4} - X_{1:4})$$

Where L_1 is a measure of location, L_2 is a measure of spread, L_3 and L_4 are used to define ratios that measure skewness and kurtosis, respectively, and $X_{(i:n)}$ denotes the ith order statistic in a sample of size n. The ratios that measure L-skewness and L-kurtosis are $\tau_3 = \frac{L3}{L2}$ and $\tau_4 = \frac{L4}{L2}$ where τ_3 is the measure of L-skewness and τ_4 is the measure of L-kurtosis. Wang (1997) defined a more general case of Hosking's L-moments, called LH moments. For example, Wang (1997) defines the measure of location as $\lambda_1^n = E[X_{(n+1):(n+1)}]$ where the expectation of the largest observation in a sample is of size $n + 1$. Hosking's L-moments are a special case of Wang's LH moments when $n = 0$.

1.2 Values of L-moments, L-skewness and L-kurtosis for some distributions

In 1990, Hosking published a paper with the values of L-moments he had developed for various distributions. Table 1 lists the L-moments for some of the distributions.

Table 1: THEORETICAL VALUES FOR L-MOMENTS

distribution	L1	L2	L3	L4	$\tau3$	$\tau4$
Normal	0	$1/\sqrt{\pi}$	0	$0.1226/\sqrt{\pi}$	0	0.1226
Uniform	1/2	1/6	0	0	0	0
Exponential	1	1/2	1/6	1/12	1/3	1/6
Gumbel	0.5772	$\ln(2)$	$0.1699\ln(2)$	$0.1504\ln(2)$	0.1699	0.1504
Log-normal	$e^{1/2}$	0.520499877	0.240991443	0.152506464	0.463	0.293

The values of L-skewness and L-kurtosis for a larger set of distributions appeared in Hosking (1992).

1.3 Use of L-Moments

The L-moments of a real-valued random variable X exists if and only if X has finite mean. A distribution may be specified by its L-moments even if some of its classical moments do not exist [5]. The commonly cited advantages of L-moments over classical moments are: able to characterize a wider range of distributions; more robust to the presence of outliers in the data when estimated from a sample; and less subject to bias in estimation and approximate their asymptotic normal distribution more closely. An example of a distribution that is characterized by L-moments but not by classical moments is the t-student distribution. The classical moments do not exist for the mean when v, the degrees of freedom, are less than two and the variance when v are less than three. However, the L-moments exist for the t-student distribution with $v = 2$. Section 1.6 gives an example of how L-moments are more robust to the presence of outliers in the data when estimated from a sample.

Currently L-moments are being used instead of classical moments to characterize distributions in the fields of Water Resources, Climate studies, Astronomy, and Hydrology. In publications on these fields, point estimations of L-moments have been calculated for real data. When using L-moments to estimate the parameters of the model, L-moments gave a better approximation of the data compared to classical moments. So far in these fields they have not estimated L-moments using confidence intervals.

Fill and Stedinger (1995) used an L-moment test developed by Hosking (1985) which is based on the shape parameter, κ, of the generalized extreme value (GEV) distribution. The L-moment test was performed on the Gumbel distribution, or extreme value distribution, to model flood flows and extreme rainfall depths. The study showed L-moments were useful for goodness-of-fit tests and distribution selection.

Waylen and Zorn (1998) estimated parameters using L-moments and used them to estimate the return periods of various water flows using the log-normal distribution. The study used the log-normal distribution to model and predict the mean and annual flows for five test sites in north central Florida. By estimating L-skewness and L-kurtosis they found that the log-normal distribution was the appropriate model for predicting the mean and annual flows in Florida.

Gingras, Adamowski, and Pilon (1994) used nine weighted regional values of L-moments computed from 183 natural flow stations from Ontario and Quebec with a record length of at least twenty years to determine the use of nonparametric methods in regional analysis. They conducted a homogeneity test of the data set to determine if the data came from the same probability distribution. By using the L-moments homogeneity test they concluded that smaller regions were more homogeneous than the entire data set.

1.4 Estimation of L-moments from a sample

L-moments are estimated from samples using functions of weighted means of order statistics. The L-moments and ratios of L-moments are estimated by

$$l_1 = \bar{x}$$

$$l_2 = 2w_2 - l_1$$

$$l_3 = 6w_3 - 6w_2 + l_1$$

$$l_4 = 20w_4 - 30w_3 + 12w_2 - l_1$$

$$\tau_3 = l_3/l_2$$

$$\tau_4 = l_4/l_2$$

where

$$w_2 = \frac{1}{n(n-1)} \sum_{i=2}^{n} (i-1) x_{i:n}$$

$$w_3 = \frac{1}{n(n-1)(n-2)} \sum_{i=3}^{n} (i-1)(i-2) x_{i:n}$$

$$w_4 = \frac{1}{n(n-1)(n-2)(n-3)} \sum_{i=4}^{n} (i-1)(i-2)(i-3) x_{i:n}$$

and $\bar{x}$ is the sample mean [10].

In order to make programming easier these expressions can be rewritten [11]. L-skewness can be rewritten when $n > 2$,

$$\sum_{i=1}^{n} \frac{c_i x_{(i)} + \bar{x}/n}{L_2},$$

where

$$c_i = 6\frac{(i-1)(i-2)}{n(n-1)(n-2)} - 6\frac{(i-1)}{n(n-1)}.$$

L-kurtosis can be rewritten for $n > 3$,

$$\sum_{i=1}^{n} \frac{d_i x_{(i)} - \bar{x}/n}{L_2},$$

where

$$d_i = 20\frac{(i-1)(i-2)(i-3)}{n(n-1)(n-2)(n-3)} - 30\frac{(i-1)(i-2)}{n(n-1)(n-2)} + 12\frac{(i-1)(i-2)(i-3)}{n(n-1)(n-2)(n-3)}.$$

Hosking prepared L-moments as a package of Fortran subroutines for the calculation of L-moments and their use in regional frequency analysis. L-moments is available through StatLib. The Department of Statistics at Carnegie Mellon has a depository of software and data sets. L-moments can be accessed through StatLib or directly from the L-moments web page residing at IBM. The web address is http://www.research.ibm.com/people/h/hosking/lmoments.html. StatLib which gives insight into information about upcoming statistical meetings, software, and datasets. StatLib distributes statistical software packages as well as gives interesting datasets from various sources. The program used for simulations in this work was specially prepared in Gauss.

1.5 Parametric Confidence Intervals for L-moments

Hosking (1990) obtained an asymptotic result specifying the sample L-moments have a multivariate normal distribution as $n \to \infty$. The standard deviations of the estimates depend however on the distribution of the variable. So in order to be able to build confidence intervals we would need to know the distribution of the variable.

Since L-moments have a multivariate normal distribution as $n \to \infty$, we can use the confidence interval formula of a normal distribution. Our confidence interval has the form estimate ± margin of error where the margin of error is the product of the critical value from the sampling distribution of the estimator and the standard error. Hence we have $\hat{\theta} \pm z_{\alpha/2} \times SE$ where the standard error, SE, depends on the distribution of x.

Hosking's results are based on the asymptotic theory for linear combinations of order statistics.

1.6 Examples Using L-moments

To get a better understanding of how ratios of L-moments measure the shape of a distribution better than classical moments, two data sets one with an outlier and one that is bimodal have been chosen. The goal is to compare L-moments to classical moments.

The first example is a data set for verbal SAT scores. The histogram for the data is given in figure 1. Notice that the histogram is roughly symmetric with an outlier, which is apparent when looking at the boxplot. The L-moments and classical moments have been estimated with a program written in Minitab. The results are given in table 2.

Table 2: COMPARING L-MOMENTS AND CLASSICAL MOMENTS FOR DATA WITH AN OUTLIER

type	L-moments		Classical moments	
with outlier	$\tau 3$	-0.00275948	$b1$	-0.194410
	$\tau 4$	0.132499	$b2$	3.09620
without outlier	$\tau 3$	0.0259907	$b1$	0.0302307
	$\tau 4$	0.112676	$b2$	2.60699

The frequency distribution of the sample of size 100 is slightly skewed to the left with one outlier which is apparent when looking at the boxplot. If we do not consider the outlier the distribution is fairly symmetric. The skewness as measured by L-moments is 0.0259907 and classical moments is 0.0302307. If we add the outlier our measures of skewness by L-moments is -0.00275948 and by classical moments is -0.194410. The measure for classical moments gives a value of a more skewed distribution once the outlier is added. Thus classical skewness is more sensitive to outliers than L-skewness. Therefore L-skewness is more robust to the presence of outliers than classical skewness.

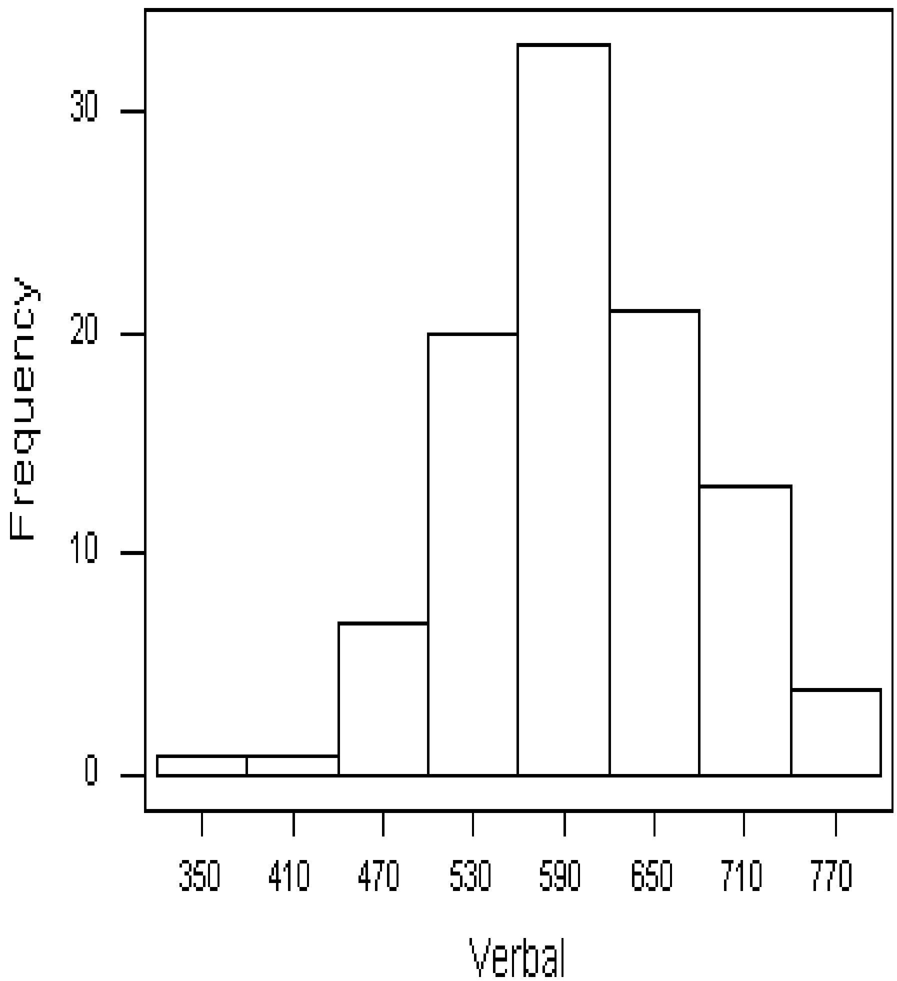

Figure 1: SAT VERBAL SCORES

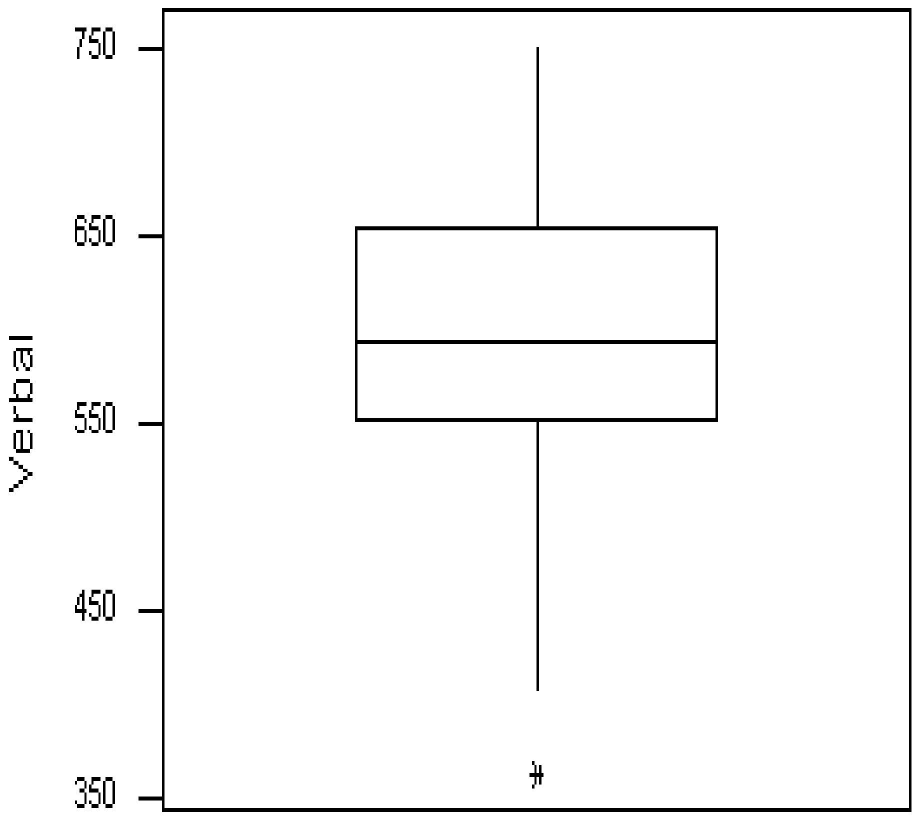

Figure 2: THE SAT VERBAL SCORES

The next data set deals with the length of eruptions for the geyser at Old Faithful. The histogram for the data, given in figure 2, is bimodal. The results for L-moments and classical moments are given in table 3.

Table 3: COMPARING L-MOMENTS AND CLASSICAL MOMENTS USING OLD FAITHFUL DATA

L-moments		Classical moments	
$\tau 3$	-0.184660	$b1$	-0.606375
$\tau 4$	-0.00586429	$b2$	1.83726

The distribution of the length of eruptions is bimodal so we would expect a kurtosis value smaller than that of the uniform distribution. For the uniform distribution $\tau 4 = 0$ and $b2 = 1.8$. But the bimodal distribution gives $\tau 4 = -0.00586429 < 0$ and $b2 = 1.83726 > 1.8$. Thus in this case L-kurtosis gives a better representation of the bimodality than classical kurtosis.

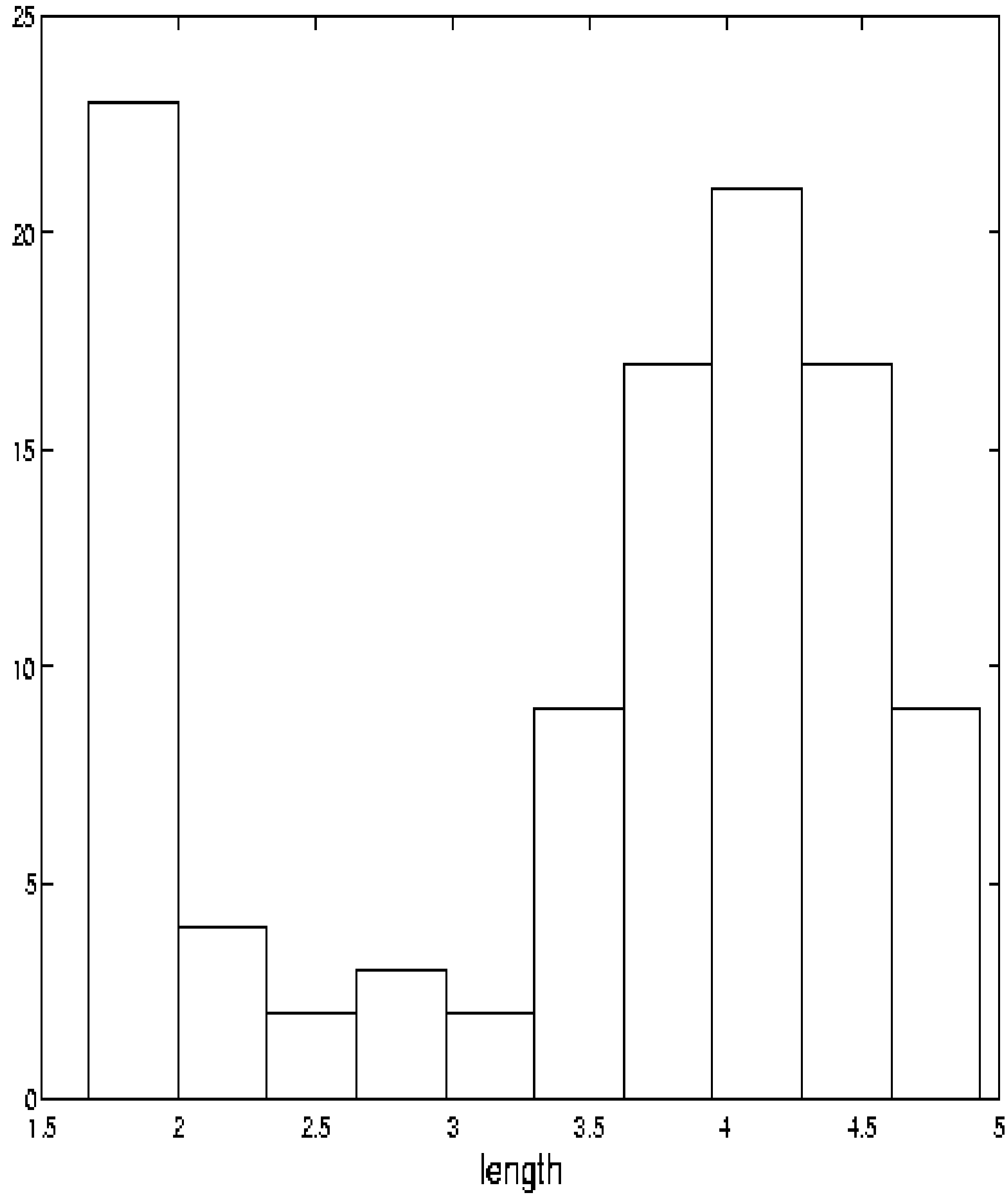

Figure 3: LENGTH OF ERUPTIONS OF OLD FAITHFUL

The following program, written in Minitab, was used to calculate the L-moments and classical moments.

```
name c1 'x' c2 'i' c3 'x(i)' c4 'w2' c5 'w3' c6 'w4' c7 's wei'
name c8 'K wei' k5 'b1' k6 'b2' c11 'z' c12 'z3' c13 'z4'
name k1 'n' k2 'L2' k3 'tau3' k4 'tau4'
count c1 k1
set c2
1:k1
end
sort c1 c3
let c4=(c2 − 1)/(k1*(k1 − 1))
let c5=(c2 − 1)*(c2 − 2)/(k1*(k1 − 1)*(k1 − 2))
let c6=((c2 − 1)*(c2 − 2)*(c2 − 3))/(k1*(k1 − 1)*(k1 − 2)*(k1 − 3))
let c7=6*c5 − 6*c4
let k2=sum(2*c4*c3)-mean(c1)
let c8=20*c6 − 30*c5+12*c4
let c9=c7*c3
let k3=(sum(c9)+mean(c1))/k2
let c10=c8*c3
let k4=(sum(c10)-mean(c1))/k2
let c11=(c1-mean(c1))/std(c1)
let c12=(c11)**3
let k5=mean(c12)
let c13=(c11)**4
let k6=mean(c13)
print k3 k4
print k5 k6
```

CHAPTER 2

BOOTSTRAPPING

Bootstrapping, a method developed by Efron in 1979, is a resampling method that takes subsamples of size n with replacement from a sample of size n. The idea is to use the empirical sampling distribution obtained with the subsamples as a substitute of the true sampling distribution of the statistic, which we ignore. The usual number of subsamples is 1000. The most common application of bootstrapping is building confidence intervals without knowing the distribution of the statistic. Besides bootstrapping there are other resampling methods used to resample data of size n. Another type of resampling method is jackknifing which was developed by Tukey in 1958. Jackknifing, which is similar to bootstrapping, systematically takes subsamples of size $n-1$ with replacement from a sample of size n leaving out one observation each time. All possible samples of size $n-1$ are used and for each subsample the statistics are computed.

2.1 Bootstrap Confidence Intervals

Bootstrap confidence intervals provide a good approximation to the exact confidence interval for many distributions. There are several ways of building confidence intervals for distributions using bootstrapping results. The easier methods are the standard interval, first percentile (Efron), and the second percentile (Hall).

The standard interval method, which assumes a normal asymptotic distribution for the statistic, builds confidence intervals using bootstrap estimates for the standard deviation of the statistic. The bootstrap standard deviation is the standard deviation of the values of the statistic $\hat{\theta}$ in all the subsamples. If we assume a normal distribution for $\hat{\theta}$ the $1-\alpha$ confidence interval for θ can be written as $\hat{\theta}-z_{\alpha/2}\hat{\sigma}_B, \hat{\theta}+z_{\alpha/2}\hat{\sigma}_B$, where $\hat{\sigma}_B$ is the estimated bootstrap standard deviation. The requirements necessary for the standard interval method to work efficiently are: $\hat{\theta}$ must have an approximately normal distribution; $\hat{\theta}$ must be unbiased in order to have reliable results about the mean value for repeated samples from the population of interest, θ; and bootstrap resampling must give us a good approximation to σ. Although the standard bootstrap confidence interval requires only 100 bootstrap subsamples to be taken to find a good estimate of the standard deviation of an estimator, other bootstrap confidence intervals require a larger number of bootstrap subsamples.

The first percentile and second percentile methods both work with using percentiles from a bootstrapped distribution to approximate the percentiles of the distribution of an estimator. Unlike the standard interval, the first and second percentile methods do not make assumptions about the distribution of the estimator. The way the first and second percentile methods are found are quite similar. Once the original sample has been bootstrapped and sorted, the first percentile method locates the two values that contain the middle $100(1-\alpha)\%$ of estimates.

After the original sample has been bootstrapped and sorted, the second percentile method looks at the difference in errors between the bootstrap estimate, $\hat{\theta}_B$, and the estimate of θ from the original sample, $\hat{\theta}$. Thus the formula $\epsilon_B = \hat{\theta}_B - \hat{\theta}$ is used to approximate the errors of the distribution for $\hat{\theta}$. Once ϵ_B is found, we use the limits ϵ_L and ϵ_H from the bootstrap distribution where $\epsilon_L = \hat{\theta}_L - \theta$ is the $1-\alpha/2$ probability and $\epsilon_H = \hat{\theta}_H - \theta$ is the $\alpha/2$ probability. The limits of ϵ_L and ϵ_H are the sampling errors of the errors of the limits between $100(1-\alpha)\%$. Thus the $100(1-\alpha)\%$ confidence limits for θ are $\hat{\theta}-\epsilon_H < \theta < \hat{\theta}-\epsilon_L$. The confidence interval for the second percentile is given as $\text{Prob}(2\hat{\theta} - \hat{\theta}_H < \theta < 2\hat{\theta} - \hat{\theta}_L) = 1 - \alpha$. When working with a skewed bootstrap distribution the first and second percentile methods will behave differently. Unfortunately it is not possible to determine which method is best to use. As mentioned earlier the calculation of bootstrap confidence intervals for the first and second percentiles require more bootstrap samples than the standard confidence interval. This is necessary since we need to accurately estimate the percentage points for the bootstrap distribution. Thus using 1000 bootstrap subsamples give us more accurate results for both the first and second percentile methods.

2.2 An Example Comparing Confidence Intervals

When we know the distribution of the statistic the results obtained by classical statistical theory and by bootstrapping are quite similar. To show this a program written in Gauss to calculate the three simple bootstrap confidence intervals for sample data was used. The data set selected is roughly normal with a sample size of 50. The 95% confidence interval for the sample was calculated by using the formula $\bar{x} \pm t^* \frac{s}{\sqrt{n}}$. The confidence interval for sample was found to be $9.911 \pm 2.007 \frac{1.928}{\sqrt{50}}$ or $(9.36, 10.46)$. When calculating the bootstrap confidence intervals for the sample data, the standard confidence interval formula $\bar{x} \pm z^* s_B$, where s_B is the standard deviation of all the sample means of the subsamples, gave $9.911 \pm (1.96)0.2742$ or a confidence interval of $(9.37, 10.45)$. For the first percentile method, the values that exceeded the 2.5% and 97.5% of the generated distribution were found. Those values were 9.37 and

10.45 which gives a 95% confidence interval of $(9.37, 10.45)$. The second percentile method calculated the difference between the bootstrap mean and the sample mean. This gave a 95% confidence interval of $(9.37, 10.45)$. After analyzing the results it is obvious that the standard interval and the percentile methods give confidence intervals very similar to the original data's confidence interval. Thus in this particular situation bootstrapping the sample data approximates the sampling distribution well.

CHAPTER 3

PERFORMANCE OF BOOTSTRAP CONFIDENCE INTERVALS

In order to build confidence intervals Hosking's results require that we know the distribution of the variable in order to find the standard deviation of the estimates of L-moments. It would be nice to have a "distribution free" confidence interval. Bootstrapping is a useful resampling method that gives us information about an unknown sampling distribution. By bootstrapping we have a good approximation about what the sampling distribution looks like. Therefore we can build distribution free confidence intervals using bootstrapping results.

3.1 Calculation of Bootstrap Confidence Intervals for L-moments

There are several ways of building confidence intervals using bootstrapping results. The three most simple ones that were mentioned earlier are the standard interval, first percentile, and second percentile. The way we calculate each of the confidence intervals are given below.

When calculating the standard confidence interval we first calculate the standard deviation, σ_B, of all values of the statistic (considering all 1000 subsamples). Once our standard deviations are calculated we assume a normal distribution for the statistic and the confidence interval is defined as the point estimate $\pm z^* \sigma_B$.

To find a 95% confidence interval for the first percentile we must calculate the value of the statistic for each subsamples, order them, then take the value that exceeds 2.5% of the generated distribution and the value that exceeds 97.5% of the generated distribution.

The second percentile calculates the difference between the bootstrap estimate, $\hat{\theta}_B$, and the estimate of θ from the original sample, $\hat{\theta}$, giving the formula $\epsilon_B = \hat{\theta}_B - \hat{\theta}$. This is then assumed to approximate the distribution of errors for $\hat{\theta}$, where ϵ_B is used to find the limits ϵ_L and ϵ_H such that $\hat{\theta} - \epsilon_H < \theta < \hat{\theta} - \epsilon_L$. In this case $\hat{\theta}$ is either L1, L2, L3, L4, $\tau 3$, or $\tau 4$.

3.2 Calculation of Empirical Coverage Through Simulations

To determine how well L-moments and ratios of L-moments behave, I wrote a program using Gauss, a mathematical software package, to compute the confidence intervals and average widths for the normal, uniform, gumbel, log-normal, and exponential distributions with sample sizes of 10, 20, 30, 40, and 50. The theoretical values for the L-moments and ratios of L-moments of these distributions were given in Hosking's paper.

The first step of the program was to determine the number of bootstrap subsamples to generate. One thousand bootstrap subsamples were used since the percentile methods require a larger number of subsamples in order to obtain a better approximation to the original data. There were 10000 replications taken in order to get a good approximation of the original sample. The theoretical values were then given for each of the distributions. The program then ran a loop of commands that generated the data for the given distribution. The sample mean was then calculated. The program then calculated the weights for L-skewness and L-kurtosis and their values from the original sample. Storage space was cleared for the subsamples. Once these steps were performed the original sample was bootstrapped. The sample mean for each subsample was calculated then the mean and standard deviation of the means of the subsample were calculated. Another loop was created to calculate the L-moments for each subsample. From this the mean and standard deviation of the L-moments of the subsamples was found. Once these steps were completed the three simple bootstrap confidence intervals were calculated for each of the L-moments and ratios of L-moments.

To determine the standard interval, the normal distribution was assumed and the normal confidence interval was used. For each of the L-moments and ratios of L-moments the low and high values of the confidence interval were found. For the first percentile the values of the bootstrapped estimates were sorted for each of the L-moments and ratios of L-moments. To find the 95% confidence intervals for the first percentile method the value that exceeds 2.5% and 97.5% of the sorted subsamples were found for each of the L-moments and ratios of L-moments. The second percentile method took the difference between the bootstrapped L-moments and the L-moments of the original sample. The differences were then sorted for each of the L-moments and ratios of L-moments. From each of the sorted differences the value that exceeds 2.5% and 97.5%, the lower and upper errors, were found. Finally for each of the confidence intervals the nominal 95% was found by determining whether each of the lower and upper values were greater than or less than the theoretical values. If a value was less than or greater than the theoretical value then a counter was used to keep track of all of the values outside of the range. When printing the final results the nominal 95% confidence intervals were obtained by first subtracting one from the values outside of the theoretical value range then dividing by the number of repetitions. This was then multiplied by 100 to get each of the three confidence intervals.

To calculate the average widths storage space was reserved for each of the L-moments and ratios of L-moments. Then for each of the confidence intervals the difference between the upper and lower bounds for each interval was calculated and added to the value of the average width in storage. Finally the last average width stored was then divided by the number of repetitions.

3.3 Description of the Research

The research question dealt with in this work was how well do bootstrap confidence intervals behave in terms of coverage and average width for estimating L-moments and ratios of L-moments? Since Hosking's results about the normality of the estimators of L-moments are based on an asymptotic approximation, we are particularly interested in knowing how well bootstrap confidence intervals behave for small sample sizes. A 95% confidence interval was used to calculate how the normal, uniform, gumbel, exponential, and log-normal distributions behave when using bootstrapping techniques with samples of size 10, 20, 30, 40, and 50. Since the normal and uniform distributions are symmetric more interest was emphasized on how well bootstrap confidence intervals behaved for skewed distributions. Thus the gumbel, exponential, and log-normal distributions hold more interest than the symmetric distributions.

3.4 Empirical Coverage

The computed nominal 95% coverage is based on 1000 bootstrap subsamples with 10000 replications for the normal, uniform, gumbel, exponential, and log-normal distributions. Each table compares each of the L-moments and ratios of L-moments with the sample sizes of 10, 20, 30, 40, and 50.

Table 4: THE COMPUTED NOMINAL 95% COVERAGE FOR L1

n=10		Normal	Uniform	Gumbel	Exponential	Log-normal
	Standard Interval	90.6	90.5	89.9	85.4	79.2
	First Percentile	90.1	91.4	89.7	85.8	80.2
	Second Percentile	90.4	89.0	89.2	83.6	77.0
n=20	Standard Interval	93.0	92.7	91.7	89.5	84.4
	First Percentile	92.6	93.2	91.7	89.9	85.4
	Second Percentile	92.7	91.9	91.4	88.4	82.4
n=30	Standard Interval	93.6	93.9	92.5	90.8	86.7
	First Percentile	93.5	94.2	92.6	91.1	87.3
	Second Percentile	93.6	93.4	92.3	89.7	85.0
n=40	Standard Interval	93.9	94.2	93.9	92.7	87.9
	First Percentile	93.9	94.5	93.8	92.9	88.4
	Second Percentile	93.7	93.8	93.6	92.0	86.1
n=50	Standard Interval	94.4	94.4	93.6	92.8	88.8
	First Percentile	94.4	94.6	93.5	93.0	89.6
	Second Percentile	94.3	94.1	93.3	91.8	87.5

When the distribution is symmetric or moderately skewed, all methods work in a similar way. For the more skewed distributions the first percentile method works a little better and for highly skewed distributions the first percentile method works better than the second percentile method and even better than the standard interval, since the standard interval assumes normality for the sampling distribution, and when the distribution of the variable is highly skewed a larger sample is necessary for $\bar{x}$ to be normal.

Table 5: THE COMPUTED NOMINAL 95% COVERAGE FOR L2

n=10		Normal	Uniform	Gumbel	Exponential	Log-normal
	Standard Interval	83.6	89.5	78.0	72.8	83.7
	First Percentile	80.7	86.4	75.0	69.5	84.4
	Second Percentile	81.9	79.9	80.2	72.9	65.2
n=20	Standard Interval	88.8	93.1	84.0	81.1	86.9
	First Percentile	88.2	91.9	83.6	80.2	84.5
	Second Percentile	88.4	87.2	86.7	82.4	74.0
n=30	Standard Interval	90.5	94.4	86.0	84.6	83.1
	First Percentile	90.2	93.9	85.8	84.4	77.3
	Second Percentile	90.8	89.4	88.6	86.3	73.6
n=40	Standard Interval	91.2	94.3	88.2	87.1	77.2
	First Percentile	91.2	93.9	88.3	87.2	68.0
	Second Percentile	92.0	89.8	90.2	88.0	70.3
n=50	Standard Interval	91.8	94.5	89.3	87.8	69.6
	First Percentile	91.8	94.3	89.5	88.0	58.0
	Second Percentile	92.5	91.1	91.4	89.3	64.4

When the distribution is symmetric all methods work in a similar way except the second percentile method under covers the smaller sample sizes of the uniform distribution. For the moderately skewed distributions the three methods worked in a similar way but did not approximate the nominal coverage as well as the symmetric distributions. The second percentile method worked the best for these distributions. The highly skewed distribution performed peculiar once it reached a sample of size 30. This was true for all of the methods.

Table 6: THE COMPUTED NOMINAL 95% COVERAGE FOR L3

n=10		Normal	Uniform	Gumbel	Exponential	Log-normal
	Standard Interval	99.8	99.8	91.9	78.1	80.5
	First Percentile	99.9	100.0	92.4	74.0	76.9
	Second Percentile	79.7	80.7	74.1	67.0	52.4
n=20	Standard Interval	97.0	96.9	89.3	79.8	85.2
	First Percentile	98.0	98.2	89.1	77.4	85.0
	Second Percentile	86.8	86.4	82.7	77.2	64.3
n=30	Standard Interval	96.1	95.7	88.5	82.4	87.4
	First Percentile	96.7	97.2	88.1	81.3	88.0
	Second Percentile	89.4	88.8	85.1	82.1	71.6
n=40	Standard Interval	95.4	95.5	89.3	83.4	88.7
	First Percentile	95.7	96.5	89.1	82.7	88.1
	Second Percentile	90.7	90.3	87.3	85.1	75.8
n=50	Standard Interval	95.3	95.3	90.0	84.5	89.0
	First Percentile	95.4	96.2	89.9	84.0	87.5
	Second Percentile	91.8	91.0	88.7	85.8	77.6

When the distribution is symmetric the standard interval method has a coverage closer to the nominal coverage. For the moderately and highly skewed distributions the standard interval and the first percentile work in a similar way. However the second percentile poorly approximates the distributions especially when the sample size is small. Therefore it is not recommended to use the second percentile for finding the nominal coverage for L3.

Table 7: THE COMPUTED NOMINAL 95% COVERAGE FOR L4

n=10		Normal	Uniform	Gumbel	Exponential	Log-normal
	Standard Interval	99.5	100.0	97.3	91.0	84.8
	First Percentile	99.6	100.0	97.7	91.1	83.5
	Second Percentile	79.6	82.6	75.8	70.3	59.7
n=20	Standard Interval	95.6	99.8	89.4	83.8	83.9
	First Percentile	95.9	99.9	88.7	82.1	81.5
	Second Percentile	82.0	87.6	77.6	72.0	54.8
n=30	Standard Interval	94.2	98.9	87.1	83.6	86.1
	First Percentile	94.0	99.5	86.0	81.6	85.0
	Second Percentile	85.5	90.3	80.9	78.1	60.0
n=40	Standard Interval	93.6	98.1	87.1	83.7	86.8
	First Percentile	93.4	98.7	86.3	81.9	86.9
	Second Percentile	88.0	90.9	84.0	81.0	66.2
n=50	Standard Interval	93.4	97.5	87.1	83.6	88.3
	First Percentile	93.2	98.1	86.4	82.5	88.6
	Second Percentile	88.9	91.7	86.1	83.1	70.4

When the distribution is symmetric, moderately skewed, or highly skewed, the standard interval and the first percentile methods work in a similar way. However the second percentile method under covers the distributions until it reaches a sample size of 40 for the moderately skewed distributions.

Table 8: THE COMPUTED NOMINAL 95% COVERAGE FOR $\tau 3$

n=10		Normal	Uniform	Gumbel	Exponential	Log-normal
	Standard Interval	99.6	99.7	98.4	97.4	91.9
	First Percentile	99.9	100.0	99.1	97.5	89.9
	Second Percentile	90.0	92.6	87.3	86.9	82.8
n=20	Standard Interval	97.0	97.4	95.2	94.2	85.8
	First Percentile	98.0	98.2	95.5	93.4	81.5
	Second Percentile	90.2	92.9	87.4	85.7	79.0
n=30	Standard Interval	96.0	96.4	93.6	93.8	84.0
	First Percentile	96.7	97.2	93.8	93.2	79.7
	Second Percentile	90.9	93.5	87.5	87.3	78.0
n=40	Standard Interval	95.1	96.1	93.1	93.1	83.8
	First Percentile	95.7	96.5	93.1	92.7	80.4
	Second Percentile	91.5	94.0	88.7	88.7	78.6
n=50	Standard Interval	95.0	95.8	92.9	93.3	83.6
	First Percentile	95.4	96.2	93.0	92.6	80.6
	Second Percentile	92.3	93.9	89.5	89.5	80.0

When the distribution is symmetric or moderately skewed, the standard interval and the first percentile methods work in a similar way. The second percentile does not reach the nominal coverage even when the sample size reaches 50. The highly skewed distribution, which gives the worst results, gives the best coverage with the standard interval method. Therefore another bootstrap method with correction for bias should be used for highly skewed distributions.

Table 9: THE COMPUTED NOMINAL 95% COVERAGE FOR $\tau 4$

n=10		Normal	Uniform	Gumbel	Exponential	Log-normal
	Standard Interval	100.0	100.0	100.0	100.0	95.7
	First Percentile	100.0	100.0	100.0	100.0	99.2
	Second Percentile	89.1	88.9	85.1	80.9	74.8
n=20	Standard Interval	98.1	99.9	96.6	94.6	84.5
	First Percentile	99.2	99.9	98.2	96.5	86.0
	Second Percentile	89.7	91.4	86.1	82.1	75.0
n=30	Standard Interval	96.7	99.2	94.1	93.2	81.8
	First Percentile	97.7	99.5	95.4	94.7	81.5
	Second Percentile	90.5	92.4	86.6	84.1	74.0
n=40	Standard Interval	95.8	98.5	93.4	92.4	81.1
	First Percentile	96.7	98.7	94.5	93.2	80.2
	Second Percentile	91.1	93.0	87.9	84.7	74.2
n=50	Standard Interval	95.7	97.6	93.2	92.0	80.5
	First Percentile	96.3	98.1	93.8	92.8	79.5
	Second Percentile	91.8	93.2	88.5	86.1	75.8

When the distribution is symmetric or moderately skewed, the standard interval and the first percentile methods work in a similar way. However the first percentile method works slightly better than the standard interval for the moderately skewed distributions. For the highly skewed distribution the standard interval anf the first percentile method works in a similar way. Again, the second percentile method gives the worst coverage of all the methods.

3.5 Average Width of Confidence Intervals

The computed average width is based on 1000 bootstrap subsamples for the normal, uniform, gumbel, exponential, and log-normal distributions. Each table compares each L-moment with the sample size and distribution.

Table 10: THE AVERAGE WIDTH FOR L1

n=10		Normal	Uniform	Gumbel	Exponential	Log-normal
	Standard Interval	1.1498	0.3358	1.4401	1.0792	1.9959
	First Percentile	1.1442	0.3338	1.4271	1.0610	1.9245
	Second Percentile	1.1442	0.3338	1.4271	1.0610	1.9245
n=20	Standard Interval	0.8456	0.2451	1.0683	0.8206	1.5748
	First Percentile	0.8439	0.2444	1.0640	0.8136	1.5397
	Second Percentile	0.8439	0.2444	1.0640	0.8136	1.5397
n=30	Standard Interval	0.6975	0.2026	0.8866	0.6852	1.3440
	First Percentile	0.6969	0.2022	0.8840	0.6813	1.3203
	Second Percentile	0.6969	0.2022	0.8840	0.6813	1.3203
n=40	Standard Interval	0.6080	0.1763	0.7736	0.5986	1.1840
	First Percentile	0.6076	0.1761	0.7719	0.5961	1.1668
	Second Percentile	0.6076	0.1761	0.7719	0.5961	1.1668
n=50	Standard Interval	0.5463	0.1581	0.6943	0.5372	1.0763
	First Percentile	0.5459	0.1579	0.6933	0.5353	1.0633
	Second Percentile	0.5459	0.1579	0.6933	0.5353	1.0633

Table 11: THE AVERAGE WIDTH FOR L2

n=10		Normal	Uniform	Gumbel	Exponential	Log-normal
	Standard Interval	0.4585	0.1149	0.6222	0.5185	1.1266
	First Percentile	0.4469	0.1133	0.5959	0.4865	1.0202
	Second Percentile	0.4469	0.1133	0.5959	0.4865	1.0202
n=20	Standard Interval	0.3348	0.0743	0.4771	0.4228	0.9926
	First Percentile	0.3320	0.0739	0.4681	0.4102	0.9378
	Second Percentile	0.3320	0.0739	0.4681	0.4102	0.9378
n=30	Standard Interval	0.2776	0.0585	0.4051	0.3642	0.8888
	First Percentile	0.2762	0.0584	0.4002	0.3579	0.8532
	Second Percentile	0.2762	0.0584	0.4002	0.3579	0.8532
n=40	Standard Interval	0.2422	0.0496	0.3569	0.3240	0.8046
	First Percentile	0.2414	0.0495	0.3539	0.3197	0.7782
	Second Percentile	0.2414	0.0495	0.3539	0.3197	0.7782
n=50	Standard Interval	0.2185	0.0438	0.3226	0.2934	0.7417
	First Percentile	0.2180	0.0437	0.3205	0.2906	0.7222
	Second Percentile	0.2180	0.0437	0.3205	0.2906	0.7222

Table 12: THE AVERAGE WIDTH FOR L3

n=10		Normal	Uniform	Gumbel	Exponential	Log-normal
	Standard Interval	0.3527	0.1087	0.4569	0.3629	0.7389
	First Percentile	0.3446	0.1061	0.4353	0.3337	0.6304
	Second Percentile	0.3446	0.1061	0.4353	0.3337	0.6304
n=20	Standard Interval	0.2219	0.0673	0.3036	0.2576	0.6199
	First Percentile	0.2197	0.0667	0.2945	0.2449	0.5573
	Second Percentile	0.2197	0.0667	0.2945	0.2449	0.5573
n=30	Standard Interval	0.1763	0.0528	0.2510	0.2169	0.5736
	First Percentile	0.1751	0.0525	0.2453	0.2095	0.5294
	Second Percentile	0.1751	0.0525	0.2453	0.2095	0.5294
n=40	Standard Interval	0.1512	0.0450	0.2189	0.1943	0.5359
	First Percentile	0.1504	0.0447	0.2154	0.1890	0.5016
	Second Percentile	0.1504	0.0447	0.2154	0.1890	0.5016
n=50	Standard Interval	0.1354	0.0398	0.1976	0.1757	0.5019
	First Percentile	0.1348	0.0396	0.1949	0.1721	0.4752
	Second Percentile	0.1348	0.0396	0.1949	0.1721	0.4752

Table 13: THE AVERAGE WIDTH FOR L4

n=10		Normal	Uniform	Gumbel	Exponential	Log-normal
	Standard Interval	0.3437	0.0988	0.4508	0.3620	0.7412
	First Percentile	0.3360	0.0961	0.4332	0.3409	0.6606
	Second Percentile	0.3360	0.0961	0.4332	0.3409	0.6606
n=20	Standard Interval	0.1780	0.0500	0.2427	0.2059	0.4752
	First Percentile	0.1755	0.0496	0.2338	0.1936	0.4137
	Second Percentile	0.1755	0.0496	0.2338	0.1936	0.4137
n=30	Standard Interval	0.1326	0.0365	0.1876	0.1616	0.4184
	First Percentile	0.1313	0.0363	0.1817	0.1544	0.3709
	Second Percentile	0.1313	0.0363	0.1817	0.1544	0.3709
n=40	Standard Interval	0.1105	0.0298	0.1592	0.1407	0.3896
	First Percentile	0.1097	0.0298	0.1551	0.1351	0.3525
	Second Percentile	0.1097	0.0298	0.1551	0.1351	0.3525
n=50	Standard Interval	0.0972	0.0257	0.1421	0.1256	0.3660
	First Percentile	0.0965	0.0256	0.1389	0.1215	0.3368
	Second Percentile	0.0965	0.0256	0.1389	0.1215	0.3368

Table 14: THE AVERAGE WIDTH FOR $\tau 3$

		Normal	Uniform	Gumbel	Exponential	Log-normal
n=10	Standard Interval	0.7480	0.7731	0.7652	0.7961	0.8278
	First Percentile	0.7593	0.7780	0.7707	0.7933	0.8136
	Second Percentile	0.7593	0.7780	0.7707	0.7933	0.8136
n=20	Standard Interval	0.4255	0.4378	0.4412	0.4573	0.4946
	First Percentile	0.4276	0.4380	0.4396	0.4537	0.4812
	Second Percentile	0.4276	0.4380	0.4396	0.4537	0.4812
n=30	Standard Interval	0.3282	0.3338	0.3432	0.3511	0.3913
	First Percentile	0.3285	0.3337	0.3408	0.3485	0.3781
	Second Percentile	0.3285	0.3337	0.3408	0.3485	0.3781
n=40	Standard Interval	0.2771	0.2803	0.2908	0.2989	0.3411
	First Percentile	0.2768	0.2801	0.2889	0.2963	0.3288
	Second Percentile	0.2768	0.2801	0.2889	0.2963	0.3288
n=50	Standard Interval	0.2458	0.2461	0.2585	0.2636	0.3044
	First Percentile	0.2455	0.2459	0.2569	0.2616	0.2944
	Second Percentile	0.2455	0.2459	0.2569	0.2616	0.2944

Table 15: THE AVERAGE WIDTH FOR $\tau 4$

n=10		Normal	Uniform	Gumbel	Exponential	Log-normal
	Standard Interval	0.7314	0.7107	0.7643	0.8361	0.9186
	First Percentile	0.7235	0.6989	0.7487	0.8045	0.8659
	Second Percentile	0.7235	0.6989	0.7487	0.8045	0.8659
n=20	Standard Interval	0.3563	0.3300	0.3853	0.4448	0.5267
	First Percentile	0.3563	0.3286	0.3823	0.4374	0.5084
	Second Percentile	0.3563	0.3286	0.3823	0.4374	0.5084
n=30	Standard Interval	0.2581	0.2326	0.2847	0.3327	0.4121
	First Percentile	0.2582	0.2319	0.2826	0.3285	0.3969
	Second Percentile	0.2582	0.2319	0.2826	0.3285	0.3969
n=40	Standard Interval	0.2109	0.1875	0.2349	0.2784	0.3559
	First Percentile	0.2108	0.1870	0.2332	0.2746	0.3414
	Second Percentile	0.2108	0.1870	0.2332	0.2746	0.3414
n=50	Standard Interval	0.1827	0.1598	0.2055	0.2438	0.3177
	First Percentile	0.1826	0.1595	0.2039	0.2409	0.3053
	Second Percentile	0.1826	0.1595	0.2039	0.2409	0.3053

CONCLUSIONS

The results obtained from using the most simple bootstrap confidence intervals for symmetric distributions gave an empirical coverage very close to 95% as the sample size increased, which is what we expected to see according to Hosking (1990). However there was over coverage for the sample sizes of 10 and 20. This is likely to happen since there is more variability with smaller sample sizes. The moderately skewed distributions also gave empirical coverage of 95% as the sample size increased. However L2, L3, and L4 gave empirical coverages close to 90% as the sample size approached 50. The undercoverage of these L-moments could be due to working with a moderately skewed distribution. The log-normal distribution had the the most biased results and the worst empirical coverage of all the distributions in this work since the distribution is highly skewed. Therefore the most simple methods of bootstrap confidence intervals should not used for the log-normal distribution. Instead the bias corrected method should be used in order to obtain less bias and a better coverage.

When looking at the average widths of each distribution and sample size it should be recommended to begin finding bootstrap confidence intervals for samples of size 20 or larger. When comparing the samples of size 10 to the samples of size 20, the average widths decrease by approximately half. Thus starting with a sample size of 20 gives better coverage of the distribution.

In conclusion, it appears that bootstrapping can be used to produce confidence intervals for L-moments and ratios of L-moments. After observing the behavior of the three methods, the standard interval and the first percentile approximate the nominal 95% coverage better than the second percentile method. Thus it is recommended to work with either the standard interval or the first percentile method unless the distribution is highly skewed in which case the bias corrected bootstrap confidence interval would be more recommendable.

BIBLIOGRAPHY

[1] K. P. Balanda and H. L. MacGillivray, Kurtosis: a critical review. *American Statistician.* 42 (1998) 111–119.

[2] T. J. DiCiccio and B. Efron, Bootstrap Confidence Intervals. *Statistical Science.* 11 (1996) 189–228.

[3] H. D. Fill and J. R. Stedinger, L Moment and Probability Plot Correlation Coefficient Goodness-of-fit Tests for the Gumbel Distribution and Impact of Autocorrelation. *Water Resources Research.* 31 (1995) 225–229.

[4] R. A. Groeneveld, An Influence Function Approach to Describing the Skewness of a Distribution. *American Statistical Association.* 45 (1991) 97–102.

[5] J. R. M. Hosking, L-moments: Analysis and Estimation of Distributions using Linear Combinations of Order Statistics. *Royal Statistical Society.* 52 (1990) 105–124.

[6] J. R. M. Hosking, Moments or L-moments? An Example Comparing Two Measures of Distributional Shape. *The American Statistician.* 46 (1992) 186–189.

[7] http://www.research.ibm.com/people/h/hosking/lmoments.html.

[8] B. F. J. Manly, *Randomization, Bootstrap and Monte Carlo Methods in Biology*, Chapman and Hall, (1997).

[9] D. S. Moore, *The Basic Practice of Statistics*, Freeman and Company, New York (2000).

[10] P. Royston, Which Measures of Skewness and Kurtosis are Best? *Statistics in Medicine.* 11 (1992) 333–343.

[11] E. Seier, A Family of Skewness and Kurtosis Measures, Ph.D. Dissertation University of Wyoming, 1998.

[12] http://lib.stat.cmu.edu/

[13] A. Stuart and J. K. Ord, Kendall's Advanced Theory of Statistics Volume I Distribution Theory, Americans by Halsted Press, an imprint of John Wiley and Sons, New York (1994).

[14] Q. J. Wang, LH Moments for Statistical Analysis of Extreme Events. *Water Resources Research.* 33 (1997) 2841–2848.

[15] P. R. Waylen and M. R. Zorn, Prediction of Mean Annual Flows in North and Central Florida. *Journal of the American Water Resources Association.* 34 (1998) 149–157.

APPENDICES

APPENDIX A

PROGRAMS IN GAUSS

Program 1. REALDATA.pgm
This program calculates bootstrap confidence intervals for L-moments, L-skewness and L-kurtosis using data from a sample.

```
output file=a:bootcint.out on ;
/* program realdata.pgm */
/* this program reads a data file and calculates bootstrap */
/* confidence intervals for the population mean */
/* fixing the number of subsamples */
mboo=1000 ;
/* read the data file */
load x[]=a:thesdata.dat ;
/* calculate the sample size */
n=rows(x) ;
/* cleaning storage space for the subsamples */
y=zeros(n,mboo) ;
/* calculate the sample mean */
xm=meanc(x) ;
/* doing bootstrapping */
/* doing resampling */
who=rndu(n,mboo) ;
whos=n*who ;
whosi=ceil(whos) ;
py=submat(x,whosi,0) ;
y=reshape(py,n,mboo) ;
uno=ones(n,1) ;
/* calculating the sample mean for each subsample */
ym=meanc(y) ;
/* calculating the mean and stdv of the means of the subsamples */
ymm=meanc(ym);
stm=stdc(ym) ;

/* CALCULATING THE DIFFERENT CONFIDENCE INTERVALS */
/* THE STANDARD BOOTSTRAP CONFIDENCE INTERVAL */
stL=xm-1.96*stm;
stH= xm+1.96*stm;

/* PERCENTILE TYPE 1 INTERVAL */
/* sorting the values of the bootstrap estimates */
sortmean=sortc(ym,1) ;
```

```
/* fixing which percentiles */
k1=(mboo+1)*0.025 ;
k2=(mboo+1)*0.975 ;
/* finding the percentiles */
pc1L=sortmean[k1,.] ;
pc1H=sortmean[k2,.] ;

/* PERCENTILE TYPE 2 INTERVAL */
differr=ym-xm;
sdifferr=sortc(differr,1);
eL=differr[k1,.];
eH=differr[k2,.] ;
pc2L=xm-eH;
pc2H=xm-eL;
/* printing the results */
print " Confidence intervals for the mean or L1" ;
print "Standard " ;
print stL stH ;
print "Which percentiles? " ;
print "k1=" k1 "k2=" k2 ;
print "Percentile type 1 " ;
print pc1L pc1H ;
print "Percentile type 2 ";
print pc2L pc2H;
end ;
```

Program 2 COVERAGE.pgm
This program calculates the empirical coverage of confidence intervals for L-moments, L-skewness and L-kurtosis based on simulated samples of a given distribution.

```
output file=a:normal10.out on ;
/* program COVERAGE.pgm */
/* This program generates samples. For each one it calculates the L-moments.*/
/* It does bootstrapping for each sample in order to calculate confidence intervals for
the L-moments in order to study the coverage of bootstrap intervals for L-moments
and L-skewness and L-kurtosis.
/* Fixing the number of subsamples. */
mboo=1000 ;
rep=10000;
n=10;
sigma=1;
/* Specify the distribution and the theoretical values. */
print "Normal Distribution" ;
print "The results are based on " rep "simulations." ;
print "The sample size is" n "." ;
print "The number of subsamples in bootstrapping is" rep "." ;
/*The theoretical values for the L-moments.*/
tvalL1=0;
tvalL2=sigma/sqrt(pi);
tvalL3=0;
tvalLsk=0;
tvalLkur=0.1226;
tvalL4=tvalLkur*tvalL2;

/*Cleaning storage space for widths*/
swstL1=0;
swstL2=0;
swstL3=0;
swstL4=0;
swstLsk=0;
swstLkur=0;
swpc1L1=0;
swpc1L2=0;
swpc1L3=0;
swpc1L4=0;
swpc1Lsk=0;
```

```
swpc1Lk=0;
swpc2L1=0;
swpc2L2=0;
swpc2L3=0;
swpc2L4=0;
swpc2Lsk=0;
swpc2Lk=0;

/*cleaning storage space for counts of standard confidence intervals */
fLL1=0;
fUL1=0;
fLL2=0;
fUL2=0;
fLL3=0;
fUL3=0;
fLL4=0;
fUL4=0;
fLLsk=0;
fULsk=0;
fLLkur=0;
fULkur=0;

/* Cleaning storage space for counts of Percentile type 1 confidence intervals*/
fLL1p1=0;
fUL1p1=0;
fLL2p1=0;
fUL2p1=0;
fLL3p1=0;
fUL3p1=0;
fLL4p1=0;
fUL4p1=0;
fLLskp1=0;
fULskp1=0;
fLLkurp1=0;
fULkurp1=0;

/* Cleaning storage space for counts of Percentile type 2 confidence intervals */
fLL1p2=0;
fUL1p2=0;
fLL2p2=0;
```

```
fUL2p2=0;
fLL3p2=0;
fUL3p2=0;
fLL4p2=0;
fUL4p2=0;
fLLskp2=0;
fULskp2=0;
fLLkurp2=0;
fULkurp2=0;

/*Selects samples and calculates the coverage.*/
count=0;
do while count ¡ rep;
count=count + 1;
x=rndn(n,1);
/* Calculating the sample mean */
xm=meanc(x) ;
/*Calculating the weights for L-skewness and L-kurtosis*/
i=seqa(1,1,n);
pw2=(i-1)/(n*(n-1));
pw3=((i-1).*(i-2))/(n*(n-1)*(n-2));
pw4=((i-1).*(i-2).*(i-3))/(n*(n-1)*(n-2)*(n-3));
/*Calculating L-skewness and L-kurtosis for the original sample.*/
xo=sortc(x,1);
w2=sumc(pw2.*xo);
w3=sumc(pw3.*xo);
L2=2*w2-xm;
L3=6*w3-6*w2+xm;
Lskew=L3/L2;
w4=sumc(pw4.*xo);
L4=20*w4-30*w3+12*w2-xm;
Lkur=L4/L2;

/* Cleaning storage space for the subsamples */
y=zeros(n,mboo) ;
yL=zeros(n,mboo);

/* doing bootstrapping */
/* doing resampling */
/* We generate uniform random numbers between 0 and 1.*/
```

```
who=rndu(n,mboo) ;
/* We multiply by n in order to have numbers from 0 to n. */
whos=n*who ;
/* We round up in order to have numbers from 1 to n. */
/* Those numbers will indicate which elements of the sample go in each subsample.*/
whosi=ceil(whos) ;
/* We identify which elements in the sample are in each subsample. */
/* Each column of y is a subsample. */
py=submat(x,whosi,0) ;
y=reshape(py,n,mboo) ;
uno=ones(n,1) ;
/* Calculating the sample mean for each subsample. */
ym=meanc(y) ;
/* Calculating the mean and standard deviation of the means of the subsamples */
ymm=meanc(ym);
stm=stdc(ym) ;
/* Calculating the L-moments for each subsample. */
is=0;
do while is¡mboo;
is=is+1;
ys=sortc(y,is);
yL[.,is]=ys[.,is];
endo;

/*Calculating the L-moments for each subsample. */
w2L=sumc(pw2'yL);
w3L=sumc(pw3'yL);
L2L=2*w2L-ym;
L3L=6*w3L-6*w2L+ym;
LskewL=L3L./L2L;
w4L=sumc(pw4'yL);
L4L=20*w4L-30*w3L+12*w2L-ym;
LkurL=L4L./L2L;

/* CALCULATING THE MEAN AND STANDARD DEVIATION OF THE */
/* L-MOMENTS OF THE SUBSAMPLES. */
mL2=meanc(L2L);
mL3=meanc(L3L);
mL4=meanc(L4L);
mLskew= meanc(LskewL);
```

```
mLkur=meanc(LkurL);
stL2=stdc(L2L);
stL3=stdc(L3L);
stL4=stdc(L4L);
stLskew=stdc(LskewL);
stLkur=stdc(LkurL);

/* CALCULATING THE DIFFERENT CONFIDENCE INTERVALS */
/* THE STANDARD BOOTSTRAP CONFIDENCE INTERVAL*/
/* for the mean */
stL=ymm-1.96*stm;
stH= ymm+1.96*stm;
swstL1=(stH-stL) + swstL1;

/* for the L-moments */
stL2L=mL2-1.96*stL2;
stL2H=mL2+1.96*stL2;
swstL2=(stL2H-stL2L) + swstL2;
stL3L=mL3-1.96*stL3;
stL3H=mL3+1.96*stL3;
swstL3=(stL3H-stL3L) + swstL3;
stL4L=mL4-1.96*stL4;
stL4H=mL4+1.96*stL4;
swstL4=(stL4H-stL4L) + swstL4;

/* for L-skewness and L-kurtosis*/
stskL=mLskew-1.96*stLskew;
stskH=mLskew+1.96*stLskew;
swstLsk=(stskH-stskL) + swstLsk;
stkurL=mLkur-1.96*stLkur;
stkurH=mLkur+1.96*stLkur;
swstLkur=(stkurH-stkurL) + swstLkur;

/* PERCENTILE TYPE 1 INTERVAL */
/* Sorting the values of the bootstrap estimates. */
sortmean=sortc(ym,1) ;
sortL2L=sortc( L2L,1);
sortL3L=sortc( L3L,1);
sortL4L=sortc( L4L,1);
/* fixing which percentiles */
```

```
k1=(mboo+1)*0.025 ;
k2=(mboo+1)*0.975 ;
/* finding the percentiles for the mean*/
pc1L=sortmean[k1,.] ;
pc1H=sortmean[k2,.] ;
swpc1L1=(pc1H-pc1L) + swpc1L1;

/* for the L-moments */
pc1L2L=sortL2L[k1,.];
pc1L2H=sortL2L[k2,.];
swpc1L2=(pc1L2H-pc1L2L) + swpc1L2;
pc1L3L=sortL3L[k1,.];
pc1L3H=sortL3L[k2,.];
swpc1L3=(pc1L3H-pc1L3L) + swpc1L3;
pc1L4L=sortL4L[k1,.];
pc1L4H=sortL4L[k2,.];
swpc1L4=(pc1L4H-pc1L4L) + swpc1L4;

/* for L-skewness and L-kurtosis */
sortLskL=sortc( LskewL,1);
sortLkL=sortc( LkurL,1);
pc1skL=sortLskL[k1,.];
pc1skH=sortLskL[k2,.];
swpc1Lsk=(pc1skH-pc1skL) + swpc1Lsk;
pc1kurL=sortLkL[k1,.];
pc1kurH=sortLkL[k2,.];
swpc1Lk=(pc1kurH-pc1kurL) + swpc1Lk;

/* PERCENTILE TYPE 2 INTERVAL */
/* for the mean */
differr=ym-xm;
sdifferr=sortc(differr,1);
eL1=sdifferr[k1,.];
eH1=sdifferr[k2,.] ;
pc2L=xm-eH1;
pc2H=xm-eL1;
swpc2L1=(pc2H-pc2L) + swpc2L1;

/* for the L- moments*/
differL2=L2L-L2;
```

```
differL3=L3L-L3;
differL4=L4L-L4;
sdferL2L=sortc(differL2,1);
sdferL3L=sortc(differL3,1);
sdferL4L=sortc(differL4,1);
eL2=sdferL2L[k1,.];
eH2=sdferL2L[k2,.];
eL3=sdferL3L[k1,.];
eH3=sdferL3L[k2,.];
eL4=sdferL4L[k1,.];
eH4=sdferL4L[k2,.];
pc2L2L=L2-eH2;
pc2L2H=L2-eL2;
swpc2L2=(pc2L2H-pc2L2L) + swpc2L2;
pc2L3L=L3-eH3;
pc2L3H=L3-eL3;
swpc2L3=(pc2L3H-pc2L3L) + swpc2L3;
pc2L4L=L4-eH4;
pc2L4H=L4-eL4;
swpc2L4=(pc2L4H-pc2L4L) + swpc2L4;

/* for L-skewness and L-kurtosis */
diferLsk=LskewL-Lskew;
differLk=LkurL-Lkur;
sdferLsk=sortc(diferLsk,1);
sdiferLk=sortc(differLk,1);
eLt3=sdferLsk[k1,.];
eHt3=sdferLsk[k2,.];
eLt4=sdiferLk[k1,.];
eHt4=sdiferLk[k2,.];
pc2skL=Lskew-eHt3;
pc2skH=Lskew-eLt3;
swpc2Lsk=(pc2skH-pc2skL) + swpc2Lsk;
pc2kurL=Lkur-eHt4;
pc2kurH=Lkur-eLt4;
swpc2Lk=(pc2kurH-pc2kurL) + swpc2Lk;

/*Calculating 95% coverage*/
/* for the standard confidence interval*/
/*for L1*/
```

```
if tvalL1 < stL;
fLL1 = fLL1 + 1;
endif;
if tvalL1 > stH;
fUL1 = fUL1 + 1;
endif;
/*for L2*/
if tvalL2 < stL2L;
fLL2 = fLL2 + 1;
endif;
if tvalL2 > stL2H;
fUL2 = fUL2 + 1;
endif;
/*for L3*/
if tvalL3 < stL3L;
fLL3 = fLL3 + 1;
endif;
if tvalL3 > stL3H;
fUL3 = fUL3 + 1;
endif;
/*for L4*/
if tvalL4 < stL4L;
fLL4 = fLL4 + 1;
endif;
if tvalL4 > stL4H;
fUL4 = fUL4 + 1;
endif;
/*for skewness*/
if tvalLsk < stskL;
fLLsk = fLLsk + 1;
endif;
if tvalLsk > stskH;
fULsk = fULsk + 1;
endif;
/*for kurtosis*/
if tvalLkur < stkurL;
fLLkur = fLLkur + 1;
endif;
if tvalLkur > stkurH;
fULkur = fULkur + 1;
```

```
endif;

/* for the percentile 1 confidence interval*/
/*for L1*/
if tvalL1 < pc1L;
fLL1p1 = fLL1p1 + 1;
endif;
if tvalL1 > pc1H;
fUL1p1 = fUL1p1 + 1;
endif;
/*for L2*/
if tvalL2 < pc1L2L;
fLL2p1 = fLL2p1 + 1;
endif;
if tvalL2 > pc1L2H;
fUL2p1 = fUL2p1 + 1;
endif;
/*for L3*/
if tvalL3 < pc1L3L;
fLL3p1 = fLL3p1 + 1;
endif;
if tvalL3 > pc1L3H;
fUL3p1 = fUL3p1 + 1;
endif;
/*for L4*/
if tvalL4 < pc1L4L;
fLL4p1 = fLL4p1 + 1;
endif;
if tvalL4 > pc1L4H;
fUL4p1 = fUL4p1 + 1;
endif;
/*for skewness*/
if tvalLsk < pc1skL;
fLLskp1 = fLLskp1 + 1;
endif;
if tvalLsk > pc1skH;
fULskp1 = fULskp1 + 1;
endif;
/*for kurtosis*/
if tvalLkur < pc1kurL;
```

```
fLLkurp1 = fLLkurp1 + 1;
endif;
if tvalLkur > pc1kurH;
fULkurp1 = fULkurp1 + 1;
endif;

/* for the percentile 2 confidence interval*/
/*for L1*/
if tvalL1 < pc2L;
fLL1p2 = fLL1p2 + 1;
endif;
if tvalL1 > pc2H;
fUL1p2 = fUL1p2 + 1;
endif;
/*for L2*/
if tvalL2 < pc2L2L;
fLL2p2 = fLL2p2 + 1;
endif;
if tvalL2 > pc2L2H;
fUL2p2 = fUL2p2 + 1;
endif;
/*for L3*/
if tvalL3 < pc2L3L;
fLL3p2 = fLL3p2 + 1;
endif;
if tvalL3 > pc2L3H;
fUL3p2 = fUL3p2 + 1;
endif;
/*for L4*/
if tvalL4 < pc2L4L;
fLL4p2 = fLL4p2 + 1;
endif;
if tvalL4 > pc2L4H;
fUL4p2 = fUL4p2 + 1;
endif;
/*for skewness*/
if tvalLsk < pc2skL;
fLLskp2 = fLLskp2 + 1;
endif;
if tvalLsk > pc2skH;
```

```
fULskp2 = fULskp2 + 1;
endif;
/*for kurtosis*/
if tvalLkur < pc2kurL;
fLLkurp2 = fLLkurp2 + 1;
endif;
if tvalLkur > pc2kurH;
fULkurp2 = fULkurp2 + 1;
endif;
endo;

/* printing the results */
print ”The theoretical values for L1” tvalL1 ” L2 ” tvalL2;
print ”The theoretical values for L3” tvalL3 ” L4 ” tvalL4;
print ”Theoretical values for Lskewness” tvalLsk ;
print ” Lkurtosis” tvalLkur;
print ” Empirical coverage for the nominal 95% coverage.”;
print ” Confidence Intervals for L1 ” ;
print ” Standard confidence interval” 100*(1-(fLL1 + fUL1)/rep) ;
print ”lower error: ” 100*(fLL1/rep) ”upper error: ” 100*(fUL1/rep);
print ” Percentile type 1 confidence interval” 100*(1-(fLL1p1+ fUL1p1)/rep) ;
print ”lower error: ” 100*(fLL1p1/rep) ”upper error: ” 100*(fUL1p1/rep);
print ” Percentile type 2 confidence interval” 100*(1-(fLL1p2+ fUL1p2)/rep) ;
print ”lower error: ” 100*(fLL1p2/rep) ”upper error: ” 100*(fUL1p2/rep);
print ”——————————————” ;
print ”Confidence intervals for L2 ” ;
print ” Standard confidence interval” 100*(1-(fLL2 + fUL2)/rep) ;
print ”lower error: ” 100*(fLL2/rep) ”upper error: ” 100*(fUL2/rep);
print ” Percentile type 1 Confidence Interval” 100*(1-(fLL2p1+ fUL2p1)/rep);
print ”lower error: ” 100*(fLL2p1/rep) ”upper error: ” 100*(fUL2p1/rep);
print ” Percentile type 2 confidence interval” 100*(1-(fLL2p2+ fUL2p2)/rep);
print ”lower error: ” 100*(fLL2p2/rep) ”upper error: ” 100*(fUL2p2/rep);
print ”——————————————” ;
print ”Confidence intervals for L3 ” ;
print ” Standard confidence interval” 100*(1-(fLL3 + fUL3)/rep) ;
print ”lower error: ” 100*(fLL3/rep) ”upper error: ” 100*(fUL3/rep);
print ” Percentile type 1 confidence interval” 100*(1-(fLL3p1+ fUL3p1)/rep) ;
print ”lower error: ” 100*(fLL3p1/rep) ”upper error: ” 100*(fUL3p1/rep);
print ” Percentile type 2 confidence interval” 100*(1-(fLL3p2+ fUL3p2)/rep) ;
print ”lower error: ” 100*(fLL3p2/rep) ”upper error: ” 100*(fUL3p2/rep);
```

```
print "———————————————" ;
print " Confidence intervals for L4" ;
print " Standard confidence interval" 100*(1-(fLL4 + fUL4)/rep) ;
print "lower error: " 100*(fLL4/rep) "upper error: " 100*(fUL4/rep);
print " Percentile type 1 confidence interval" 100*(1-(fLL4p1+ fUL4p1)/rep) ;
print "lower error: " 100*(fLL4p1/rep) "upper error: " 100*(fUL4p1/rep);
print " Percentile type 2 confidence interval" 100*(1-(fLL4p2+ fUL4p2)/rep) ;
print "lower error: " 100*(fLL4p2/rep) "upper error: " 100*(fUL4p2/rep);
print "———————————————" ;
print " Confidence intervals for L-skewness tau 3 " ;
print " Standard confidence interval" 100*(1-(fLLsk + fULsk)/rep) ;
print "lower error: " 100*(fLLsk/rep) "upper error: " 100*(fULsk/rep);
print " Percentile type 1 confidence interval" 100*(1-(fLLskp1+ fULskp1)/rep) ;
print "lower error: " 100*(fLLskp1/rep) "upper error: " 100*(fULskp1/rep);
print " Percentile type 2 confidence interval" 100*(1-(fLLskp2+ fULskp2)/rep) ;
print "lower error: " 100*(fLLskp2/rep) "upper error: " 100*(fULskp2/rep);
print "———————————————" ;
print " Confidence intervals for L-kurtosis tau 4 " ;
print " Standard confidence interval" 100*(1-(fLLkur + fULkur)/rep) ;
print "lower error: " 100*(fLLkur/rep) "upper error: " 100*(fULkur/rep);
print " Percentile type 1 confidence interval" 100*(1-(fLLkurp1+ fULkurp1)/rep) ;
print "lower error: " 100*(fLLkurp1/rep) "upper error: " 100*(fULkurp1/rep);
print " Percentile type 2 confidence interval" 100*(1-(fLLkurp2+ fULkurp2)/rep) ;
print "lower error: " 100*(fLLkurp2/rep) "upper error: " 100*(fULkurp2/rep);
print "———————————————" ;
print "The average width ";
print "For L1 ";
print " Standard Percentile 1 Percentile 2 ";
print swstL1/rep swpc1L1/rep swpc2L1/rep ;
print "For L2 ";
print " Standard Percentile 1 Percentile 2 ";
print swstL2/rep swpc1L2/rep swpc2L2/rep ;
print "For L3 ";
print " Standard Percentile 1 Percentile 2 ";
print swstL3/rep swpc1L3/rep swpc2L3/rep ;
print "For L4 ";
print " Standard Percentile 1 Percentile 2 ";
print swstL4/rep swpc1L4/rep swpc2L4/rep ;
print "For L-skewness ";
print " Standard Percentile 1 Percentile 2 ";
```

```
print swstLsk/rep swpc1Lsk/rep swpc2Lsk/rep ;
print "For L-kurtosis ";
print " Standard Percentile 1 Percentile 2 ";
print swstLkur/rep swpc1Lk/rep swpc2Lk/rep ;

end ;
```

APPENDIX B
SELECTED RESULTS

Results for the empirical 95% coverage for $\tau 3$ and $\tau 4$ found by simulations using Gauss.
Normal Distribution with $n = 10$
Confidence intervals for L-skewness tau 3
Standard confidence interval 92.850000
lower error: 1.4700000 upper error: 5.6800000
Percentile type 1 confidence interval 93.010000
lower error: 0.87000000 upper error: 6.1200000
Percentile type 2 confidence interval 89.510000
lower error: 5.9200000 upper error: 4.5700000

Confidence intervals for L-kurtosis tau 4
Standard confidence interval 93.160000
lower error: 0.33000000 upper error: 6.5100000
Percentile type 1 confidence interval 93.790000
lower error: 0.16000000 upper error: 6.0500000
Percentile type 2 confidence interval 88.520000
lower error: 6.5400000 upper error: 4.9400000

Normal Distribution with $n = 20$
Confidence intervals for L-skewness tau 3
Standard confidence interval 96.970000
lower error: 1.4800000 upper error: 1.5500000
Percentile type 1 confidence interval 97.990000
lower error: 0.99000000 upper error: 1.0200000
Percentile type 2 confidence interval 90.180000
lower error: 4.7800000 upper error: 5.0400000

Confidence intervals for L-kurtosis tau 4
Standard confidence interval 98.050000
lower error: 0.00000000 upper error: 1.9500000
Percentile type 1 confidence interval 99.170000
lower error: 0.00000000 upper error: 0.83000000
Percentile type 2 confidence interval 89.730000
lower error: 5.4500000 upper error: 4.8200000

Normal Distribution with $n = 30$
Confidence intervals for L-skewness tau 3
Standard confidence interval 95.990000
lower error: 1.9200000 upper error: 2.0900000

Percentile type 1 confidence interval 96.650000
lower error: 1.5100000 upper error: 1.8400000
Percentile type 2 confidence interval 90.900000
lower error: 4.7400000 upper error: 4.3600000

Confidence intervals for L-kurtosis tau 4
Standard confidence interval 96.680000
lower error: 0.14000000 upper error: 3.1800000
Percentile type 1 confidence interval 97.690000
lower error: 0.080000000 upper error: 2.2300000
Percentile type 2 confidence interval 90.470000
lower error: 5.5400000 upper error: 3.9900000

Normal Distribution with $n = 40$
Confidence intervals for L-skewness tau 3
Standard confidence interval 95.090000
lower error: 2.2400000 upper error: 2.6700000
Percentile type 1 confidence interval 95.670000
lower error: 1.8600000 upper error: 2.4700000
Percentile type 2 confidence interval 91.480000
lower error: 4.1300000 upper error: 4.3900000

Confidence intervals for L-kurtosis tau 4
Standard confidence interval 95.750000
lower error: 0.27000000 upper error: 3.9800000
Percentile type 1 confidence interval 96.680000
lower error: 0.24000000 upper error: 3.0800000
Percentile type 2 confidence interval 91.110000
lower error: 5.0000000 upper error: 3.8900000

Normal Distribution with $n = 50$
Confidence intervals for L-skewness tau 3
Standard confidence interval 95.030000
lower error: 2.5400000 upper error: 2.4300000
Percentile type 1 confidence interval 95.400000
lower error: 2.2700000 upper error: 2.3300000
Percentile type 2 confidence interval 92.310000
lower error: 4.0100000 upper error: 3.6800000

Confidence intervals for L-kurtosis tau 4

Standard confidence interval 95.690000
lower error: 0.34000000 upper error: 3.9700000
Percentile type 1 confidence interval 96.260000
lower error: 0.27000000 upper error: 3.4700000
Percentile type 2 confidence interval 91.830000
lower error: 4.6500000 upper error: 3.5200000

Uniform Distribution with $n = 10$
Confidence intervals for L-skewness tau 3
Standard confidence interval 99.740000
lower error: 0.14000000 upper error: 0.12000000
Percentile type 1 confidence interval 99.970000
lower error: 0.010000000 upper error: 0.020000000
Percentile type 2 confidence interval 92.600000
lower error: 3.8200000 upper error: 3.5800000

Confidence intervals for L-kurtosis tau 4
Standard confidence interval 100.00000
lower error: 0.0000000 upper error: 0.0000000
Percentile type 1 confidence interval 100.00000
lower error: 0.0000000 upper error: 0.0000000
Percentile type 2 confidence interval 88.890000
lower error: 2.9800000 upper error: 8.1300000

Uniform Distribution with $n = 20$
Confidence intervals for L-skewness tau 3
Standard confidence interval 97.410000
lower error: 1.4700000 upper error: 1.1200000
Percentile type 1 confidence interval 98.160000
lower error: 1.0000000 upper error: 0.84000000
Percentile type 2 confidence interval 92.890000
lower error: 4.0300000 upper error: 3.0800000

Confidence intervals for L-kurtosis tau 4
Standard confidence interval 99.920000
lower error: 0.070000000 upper error: 0.010000000
Percentile type 1 confidence interval 99.930000
lower error: 0.070000000 upper error: 0.00000000
Percentile type 2 confidence interval 91.360000
lower error: 2.1200000 upper error: 6.5200000

Uniform Distribution with $n = 30$
Confidence intervals for L-skewness tau 3
Standard confidence interval 96.440000
lower error: 1.7100000 upper error: 1.8500000
Percentile type 1 confidence interval 97.160000
lower error: 1.3400000 upper error: 1.5000000
Percentile type 2 confidence interval 93.460000
lower error: 3.2600000 upper error: 3.2800000

Confidence intervals for L-kurtosis tau 4
Standard confidence interval 99.190000
lower error: 0.36000000 upper error: 0.45000000
Percentile type 1 confidence interval 99.510000
lower error: 0.42000000 upper error: 0.070000000
Percentile type 2 confidence interval 92.360000
lower error: 1.9500000 upper error: 5.6900000

Uniform Distribution with $n = 40$
Confidence intervals for L-skewness tau 3
Standard confidence interval 96.050000
lower error: 1.8900000 upper error: 2.0600000
Percentile type 1 confidence interval 96.510000
lower error: 1.5900000 upper error: 1.9000000
Percentile type 2 confidence interval 94.000000
lower error: 2.8200000 upper error: 3.1800000

Confidence intervals for L-kurtosis tau 4
Standard confidence interval 98.510000
lower error: 0.63000000 upper error: 0.86000000
Percentile type 1 confidence interval 98.720000
lower error: 0.91000000 upper error: 0.37000000
Percentile type 2 confidence interval 93.020000
lower error: 1.6700000 upper error: 5.3100000

Uniform Distribution with $n = 50$
Confidence intervals for L-skewness tau 3
Standard confidence interval 95.790000
lower error: 2.0000000 upper error: 2.2100000
Percentile type 1 confidence interval 96.200000

lower error: 1.7700000 upper error: 2.0300000
Percentile type 2 confidence interval 93.910000
lower error: 3.0100000 upper error: 3.0800000

Confidence intervals for L-kurtosis tau 4
Standard confidence interval 97.600000
lower error: 0.91000000 upper error: 1.4900000
Percentile type 1 confidence interval 98.080000
lower error: 1.0300000 upper error: 0.89000000
Percentile type 2 confidence interval 93.230000
lower error: 1.7600000 upper error: 5.0100000

Gumbel Distribution with $n = 10$
Confidence intervals for L-skewness tau 3
Standard confidence interval 98.420000
lower error: 0.00000000 upper error: 1.5800000
Percentile type 1 confidence interval 99.080000
lower error: 0.00000000 upper error: 0.92000000
Percentile type 2 confidence interval 87.340000
lower error: 7.9700000 upper error: 4.6900000

Confidence intervals for L-kurtosis tau 4
Standard confidence interval 100.00000
lower error: 0.00000000 upper error: 0.00000000
Percentile type 1 confidence interval 100.00000
lower error: 0.00000000 upper error: 0.00000000
Percentile type 2 confidence interval 85.120000
lower error: 6.9900000 upper error: 7.8900000

Gumbel Distribution with $n = 20$
Confidence intervals for L-skewness tau 3
Standard confidence interval 95.190000
lower error: 0.38000000 upper error: 4.4300000
Percentile type 1 confidence interval 95.520000
lower error: 0.16000000 upper error: 4.3200000
Percentile type 2 confidence interval 87.350000
lower error: 7.8400000 upper error: 4.8100000

Confidence intervals for L-kurtosis tau 4
Standard confidence interval 96.570000

lower error: 0.00000000 upper error: 3.4300000
Percentile type 1 confidence interval 98.200000
lower error: 0.00000000 upper error: 1.8000000
Percentile type 2 confidence interval 86.050000
lower error: 7.7300000 upper error: 6.2200000

Gumbel Distribution with $n = 30$
Confidence intervals for L-skewness tau 3
Standard confidence interval 93.640000
lower error: 0.98000000 upper error: 5.3800000
Percentile type 1 confidence interval 93.780000
lower error: 0.47000000 upper error: 5.7500000
Percentile type 2 confidence interval 87.530000
lower error: 7.6900000 upper error: 4.7800000

Confidence intervals for L-kurtosis tau 4
Standard confidence interval 94.080000
lower error: 0.030000000 upper error: 5.8900000
Percentile type 1 confidence interval 95.400000
lower error: 0.020000000 upper error: 4.5800000
Percentile type 2 confidence interval 86.620000
lower error: 7.7900000 upper error: 5.5900000

Gumbel Distribution with $n = 40$
Confidence intervals for L-skewness tau 3
Standard confidence interval 93.130000
lower error: 1.3900000 upper error: 5.4800000
Percentile type 1 confidence interval 93.090000
lower error: 0.92000000 upper error: 5.9900000
Percentile type 2 confidence interval 88.720000
lower error: 6.6000000 upper error: 4.6800000

Confidence intervals for L-kurtosis tau 4
Standard confidence interval 93.410000
lower error: 0.19000000 upper error: 6.4000000
Percentile type 1 confidence interval 94.530000
lower error: 0.040000000 upper error: 5.4300000
Percentile type 2 confidence interval 87.860000
lower error: 7.0200000 upper error: 5.1200000

Gumbel Distribution with $n = 50$
Confidence intervals for L-skewness tau 3
Standard confidence interval 92.850000
lower error: 1.4700000 upper error: 5.6800000
Percentile type 1 confidence interval 93.010000
lower error: 0.87000000 upper error: 6.1200000
Percentile type 2 confidence interval 89.510000
lower error: 5.9200000 upper error: 4.5700000

Confidence intervals for L-kurtosis tau 4
Standard confidence interval 93.160000
lower error: 0.33000000 upper error: 6.5100000
Percentile type 1 confidence interval 93.790000
lower error: 0.16000000 upper error: 6.0500000
Percentile type 2 confidence interval 88.520000
lower error: 6.5400000 upper error: 4.9400000

Exponential Distribution with $n = 10$
Confidence intervals for L-skewness tau 3
Standard confidence interval 97.360000
lower error: 0.0000000 upper error: 2.6400000
Percentile type 1 confidence interval 97.470000
lower error: 0.0000000 upper error: 2.5300000
Percentile type 2 confidence interval 86.940000
lower error: 10.240000 upper error: 2.8200000

Confidence intervals for L-kurtosis tau 4
Standard confidence interval 99.990000
lower error: 0.0000000 upper error: 0.010000000
Percentile type 1 confidence interval 100.00000
lower error: 0.0000000 upper error: 0.0000000
Percentile type 2 confidence interval 80.920000
lower error: 8.4900000 upper error: 10.590000

Exponential Distribution with $n = 20$
Confidence intervals for L-skewness tau 3
Standard confidence interval 94.170000
lower error: 0.11000000 upper error: 5.7200000
Percentile type 1 confidence interval 93.360000
lower error: 0.0000000 upper error: 6.6400000

Percentile type 2 confidence interval 85.680000
lower error: 10.850000 upper error: 3.4700000

Confidence intervals for L-kurtosis tau 4
Standard confidence interval 94.570000
lower error: 0.0000000 upper error: 5.4300000
Percentile type 1 confidence interval 96.450000
lower error: 0.0000000 upper error: 3.5500000
Percentile type 2 confidence interval 82.100000
lower error: 9.4100000 upper error: 8.4900000

Exponential Distribution with $n = 30$
Confidence intervals for L-skewness tau 3
Standard confidence interval 93.780000
lower error: 0.44000000 upper error: 5.7800000
Percentile type 1 confidence interval 93.220000
lower error: 0.11000000 upper error: 6.6700000
Percentile type 2 confidence interval 87.330000
lower error: 9.4800000 upper error: 3.1900000

Confidence intervals for L-kurtosis tau 4
Standard confidence interval 93.210000
lower error: 0.010000000 upper error: 6.7800000
Percentile type 1 confidence interval 94.700000
lower error: 0.00000000 upper error: 5.3000000
Percentile type 2 confidence interval 84.070000
lower error: 8.8200000 upper error: 7.1100000

Exponential Distribution with $n = 40$
Confidence intervals for L-skewness tau 3
Standard confidence interval 93.100000
lower error: 0.84000000 upper error: 6.0600000
Percentile type 1 confidence interval 92.720000
lower error: 0.41000000 upper error: 6.8700000
Percentile type 2 confidence interval 88.660000
lower error: 8.0500000 upper error: 3.2900000

Confidence intervals for L-kurtosis tau 4
Standard confidence interval 92.370000
lower error: 0.14000000 upper error: 7.4900000

Percentile type 1 confidence interval 93.220000
lower error: 0.030000000 upper error: 6.7500000
Percentile type 2 confidence interval 84.650000
lower error: 8.4000000 upper error: 6.9500000

Exponential Distribution with $n = 50$
Confidence intervals for L-skewness tau 3
Standard confidence interval 93.280000
lower error: 0.89000000 upper error: 5.8300000
Percentile type 1 confidence interval 92.610000
lower error: 0.54000000 upper error: 6.8500000
Percentile type 2 confidence interval 89.540000
lower error: 7.2100000 upper error: 3.2500000

Confidence intervals for L-kurtosis tau 4
Standard confidence interval 91.970000
lower error: 0.37000000 upper error: 7.6600000
Percentile type 1 confidence interval 92.780000
lower error: 0.11000000 upper error: 7.1100000
Percentile type 2 confidence interval 86.140000
lower error: 7.2700000 upper error: 6.5900000

Log-normal Distribution with $n = 10$
Confidence intervals for L-skewness tau 3
Standard confidence interval 91.850000
lower error: 0.0000000 upper error: 8.1500000
Percentile type 1 confidence interval 89.940000
lower error: 0.0000000 upper error: 10.060000
Percentile type 2 confidence interval 82.810000
lower error: 12.160000 upper error: 5.0300000

Confidence intervals for L-kurtosis tau 4
Standard confidence interval 95.690000
lower error: 0.0000000 upper error: 4.3100000
Percentile type 1 confidence interval 99.230000
lower error: 0.0000000 upper error: 0.77000000
Percentile type 2 confidence interval 74.820000
lower error: 11.000000 upper error: 14.180000

Log-normal Distribution with $n = 20$

Confidence intervals for L-skewness tau 3
Standard confidence interval 85.830
lower error: 0.010000 upper error: 14.160
Percentile type 1 confidence interval 81.490
lower error: 0.00000 upper error: 18.510
Percentile type 2 confidence interval 79.030
lower error: 14.410 upper error: 6.5600

Confidence intervals for L-kurtosis tau 4
Standard confidence interval 84.530
lower error: 0.00000 upper error: 15.470
Percentile type 1 confidence interval 85.990
lower error: 0.00000 upper error: 14.010
Percentile type 2 confidence interval 74.950
lower error: 12.300 upper error: 12.750

Log-normal Distribution with $n = 30$
Confidence intervals for L-skewness tau 3
Standard confidence interval 83.990000
lower error: 0.10000000 upper error: 15.910000
Percentile type 1 confidence interval 79.740000
lower error: 0.020000000 upper error: 20.240000
Percentile type 2 confidence interval 78.030000
lower error: 14.510000 upper error: 7.4600000

Confidence intervals for L-kurtosis tau 4
Standard confidence interval 81.780000
lower error: 0.00000000 upper error: 18.220000
Percentile type 1 confidence interval 81.460000
lower error: 0.00000000 upper error: 18.540000
Percentile type 2 confidence interval 74.030000
lower error: 13.780000 upper error: 12.190000

Log-normal Distribution with $n = 40$
Confidence intervals for L-skewness tau 3
Standard confidence interval 83.810000
lower error: 0.26000000 upper error: 15.930000
Percentile type 1 confidence interval 80.360000
lower error: 0.040000000 upper error: 19.600000
Percentile type 2 confidence interval 78.570000

lower error: 13.120000 upper error: 8.3100000

Confidence intervals for L-kurtosis tau 4
Standard confidence interval 81.120000
lower error: 0.030000000 upper error: 18.850000
Percentile type 1 confidence interval 80.160000
lower error: 0.00000000 upper error: 19.840000
Percentile type 2 confidence interval 74.220000
lower error: 13.430000 upper error: 12.350000

Log-normal Distribution with $n = 50$
Confidence intervals for L-skewness tau 3
Standard confidence interval 83.630000
lower error: 0.53000000 upper error: 15.840000
Percentile type 1 confidence interval 80.620000
lower error: 0.23000000 upper error: 19.150000
Percentile type 2 confidence interval 79.960000
lower error: 11.910000 upper error: 8.1300000

Confidence intervals for L-kurtosis tau 4
Standard confidence interval 80.520000
lower error: 0.10000000 upper error: 19.380000
Percentile type 1 confidence interval 79.520000
lower error: 0.020000000 upper error: 20.460000
Percentile type 2 confidence interval 75.820000
lower error: 12.300000 upper error: 11.880000

APPENDIX C
DATA SETS

The verbal SAT scores data set was used to compare L-moments and classical moments with and without an outlier.

623 454 643 585 719 693 571 646 613 655 662 585 580 648 405 506 669 558 577
487 682 565 552 567 745 610 493 571 682 600 740 593 488 526 630 586 610 695 539
490 509 667 597 662 566 597 604 519 643 606 500 460 717 592 752 695 610 620 682
524 552 703 584 550 659 585 578 533 532 708 537 635 591 552 557 599 540 752 726
630 558 646 643 606 682 565 578 488 361 560 630 666 719 669 571 520 571 539 580
629

The Old Faithful data set was used to show how a bimodal ditribution behaved for the ratios of L-moments.

4.37 4.70 1.68 1.75 4.35 1.77 4.25 4.10 4.05 1.90 4.00 4.42 1.83 1.83 3.95 4.83 3.87
1.73 3.92 3.20 4.58 3.50 3.80 3.80 1.80 1.95 1.77 4.28 4.40 4.65 4.50 3.43 2.93 4.33 4.13
3.72 2.33 4.57 3.58 3.70 4.25 3.58 3.67 1.90 4.13 4.53 4.10 4.12 4.00 4.93 3.68 1.85 3.83
1.85 3.33 3.73 1.67 4.63 1.83 2.03 2.72 4.03 1.73 3.10 4.62 1.88 3.52 3.77 3.43 2.00 3.73
4.60 4.18 4.58 3.50 4.62 4.03 1.97 4.60 4.00 3.75 4.00 4.33 1.82 1.67 3.50 4.20 4.43 1.90
4.08 4.50 1.80 3.70 2.50 2.27 2.93 4.63 4.00 1.97 3.93 4.07 4.50 2.25 4.25 4.08 3.92 4.73

This is the sample data, of size 50, used to compare the bootstrap confidence intervals with the confidence interval of the sample mean when the distribution is known.

10.1376 9.3530 8.6794 10.9980 11.1771 12.6618 11.6269 11.6682 8.5191 8.5131 9.8479 11.5055 10.0547 11.6623 12.5479 8.4192 10.3638 7.4711 12.3355 9.1774 9.9431 9.5695 9.5932 6.1815 12.3039 8.1821 7.8135 7.6215 9.9323 7.3255 11.5228 9.9319 10.6074 9.7960 12.4094 7.4765 13.1909 6.5203 6.8179 13.7299 11.0442 6.5886 8.8473 8.7331 9.6348 9.8501 9.2861 9.5342 13.7808 11.0621

VITA

Suzanne P. Glass

P. O. Box 10393 ETSU

Johnson City, TN 37614

EDUCATION:

East Tennessee State University, Johnson City, TN

Mathematics, B.S., May 1998

East Tennessee State University, Johnson City, TN

Mathematical Sciences, M.S., May 2000

PROFESSIONAL EXPERIENCE:

Graduate Assistant, East Tennessee State University

Department of Mathematics, August 1998 - May 2000