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Neighborhood-Restricted Achromatic Colorings of Graphs

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Neighborhood-Restricted Achromatic Colorings of Graphs

A thesis

presented to

the faculty of the Department of Mathematics

East Tennessee State University

In partial fulfillment

of the requirements for the degree

Master of Science in Mathematical Sciences

by

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A (closed) neighborhood-restricted $[\leq 2]$-coloring of a graph $G$ is an assignment of colors to the vertices of $G$ such that no more than two colors are assigned in any closed neighborhood. In other words, for every vertex $v$ in $G$, the vertex $v$ and its neighbors are in at most two different color classes. The $[\leq 2]$-achromatic number is defined as the maximum number of colors in any $[\leq 2]$-coloring of $G$. We study the $[\leq 2]$-achromatic number. In particular, we improve a known upper bound and characterize the extremal graphs for some other known bounds.
DEDICATION

I would like to dedicate my thesis to the two most important people in my life, Rebecca Lynn and James Dustin Jr.
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First and foremost, I would like to thank my thesis adviser, Dr. Teresa Haynes, for introducing me to, and inspiring me to pursue research in, graph theory. I would also like to thank Dr. Wyatt Desormeaux for his help with my research and writing, and his help keeping \LaTeX (of which I am forbidden to speak ill) working. I would like to thank Dr. Robert Beeler for always being available to answer questions about my classes and my research, and for advising me as an undergraduate student. I would like to thank Dr. Anant Godbole for working with me in my undergraduate research, and for his time and dedication preparing me for graduate school. I would also like to thank one of my undergraduate professors, who shall remain nameless, for inspiring me to excel at graduate school. And I would like to thank my friends, peers, coworkers in the CFAA, and fellow graduate students for for their love and support over the last four years at East Tennessee State University.
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1 INTRODUCTION

1.1 Introduction to Graph Theory

A graph $G = (V, E)$ consists of a finite vertex set, $V(G)$, and a finite edge set, $E(G)$. The order of a graph $G$, denoted $n(G)$, is the number of vertices in $G$, and the size of a graph $G$, denoted $m(G)$, is the number of edges in $G$; that is, $n(G) = |V(G)|$ and $m(G) = |E(G)|$. If $G$ is clear from the context, we generally use $V$, $E$, $m$, and $n$. Two vertices $u$ and $v$ are adjacent if there is an edge in $E$, denoted $uv \in E$, connecting $u$ and $v$. We say that the vertices $u, v \in V$ are incident with edge $uv$. Further, we consider only simple graphs where the edges of $G$ do not have a direction component and there are no instances of multiple edges connecting the same two vertices $u$ and $v$. The complement of $G$, denoted $\overline{G}$, is the graph with $V(G) = V(\overline{G})$ where two vertices are adjacent if and only if they are not adjacent in $G$. Thus, $E(\overline{G}) = E(G)$.

A Nordhaus-Gaddum type result is a result wherein there is an upper bound on the sum or product of a parameter on $G$ and $\overline{G}$. For any $v \in V$, we denote the graph formed by removing $v$ and all of its incident edges by $G - v$.

For two vertices $u, v \in V$, a $u$-$v$ walk $W$ is a sequence of vertices in $G$, beginning with $u$ and ending with $v$, such that the consecutive vertices in $W$ are adjacent in $G$. A path is a walk in which no vertex is repeated. The distance $d(u, v)$ between two vertices $u, v \in V$ is the minimum of the lengths of all $u$-$v$ paths in $G$. The maximum distance from $v$ to the other vertices of $G$ is called the eccentricity of $v$, $e(v)$; that is, $e(v) = \max\{d(u, v) | u \in V\}$. The diameter of $G$, $\text{diam}(G)$, is the maximum eccentricity among all the vertices of $G$. A graph that has a $u$-$v$ path for
all \( u, v \in V \) is a connected graph.

For a vertex \( v \in V \), the set \( N(v) = \{ u \in V \mid uv \in E \} \) is called the open neighborhood of \( v \) where \( N(v) \) is the set of all vertices adjacent to \( v \) in \( G \). Each vertex \( u \in N(v) \) is called a neighbor of \( v \). The closed neighborhood of a vertex \( v \), \( N[v] \), is the set of all vertices adjacent to \( v \) and \( v \) itself. That is, \( N[v] = N(v) \cup \{ v \} \). The open neighborhood of a set \( S \subseteq V \) is \( N(S) = \bigcup_{v \in S} N(v) \), and the closed neighborhood of a set \( S \subseteq V \) is \( N[S] = \bigcup_{v \in S} N[v] \). The degree in \( G \) of a vertex \( v \) is \( \deg_G(v) = |N(v)| \); if \( G \) is clear from the context then we use \( \deg(v) \). A vertex \( v \) with \( \deg(v) = 1 \) is called a leaf. The neighbor of a leaf is called a support vertex; a support vertex with more than one leaf neighbor is called a strong support vertex.

A path \( P_n \) is a graph with \( V = \{ v_1, v_2, \ldots, v_n \} \) and \( E = \{ v_i v_{i+1} \mid i = 1, 2, \ldots, n-1 \} \). A cycle \( C_n \) of order \( n \geq 3 \) is a graph with \( V = \{ v_1, v_2, \ldots, v_n \} \) and \( E = \{ v_i v_{i+1 \mod n} \mid i = 1, 2, \ldots, n \} \). A graph in which every two distinct vertices are adjacent is called a complete graph \( K_n \). A connected graph that contains no cycles is a tree \( T \). A star \( S_{1,n-1} \) is a tree with exactly one support vertex and \( n-1 \) leaves, that is, a star \( S_{1,n-1} \) is a tree with diameter 2. A double star \( S_{r,s} \) is a tree with diameter 3, that is, \( S_{r,s} \) has two support vertices \( u, v \in V \) such that \( uv \in E \) and \( u \) has \( r \) leaf neighbors while \( v \) has \( s \) leaf neighbors. The corona \( G \circ K_1 \), denoted \( \text{cor}(G) \), is formed from a graph \( G \) by attaching a new vertex \( v' \) adjacent to \( v \) for each \( v \in V(G) \).

A set \( S \subseteq V \) is a dominating set of \( G \) if every vertex \( v \in V \) is adjacent to a vertex in \( S \). The minimum cardinality of all possible dominating sets of \( G \) is called the domination number \( \gamma(G) \) of \( G \). A set \( S \subseteq V \) is a 2-packing set of a graph \( G \) if for every \( u, v \in S \), \( d(u, v) \geq 3 \). The 2-packing number, \( \rho(G) \), is the maximum cardinality
of all such 2-packing sets. A dominating set with cardinality $\gamma(G)$ is called a $\gamma(G)$-set, and a 2-packing set with cardinality $\rho(G)$ is called a $\rho(G)$-set. A dominating set $S$ of $G$ is called an efficient dominating set if it is also a 2-packing of $G$. It was shown by Bange et al. in [1] that if a graph $G$ has an efficient dominating set $S$, then $|S| = \gamma(G)$.

A coloring of a graph $G$ is a partitioning of the vertex set $V$ into color classes. A proper coloring of the vertices of a graph $G$ assigns a color to each vertex of $G$ in such a way that no two adjacent vertices have the same color. The chromatic number $\chi(G)$ is the minimum number of colors required in any proper coloring of $G$. Similarly, a proper achromatic coloring of a graph $G$ assigns colors to each vertex of $G$ such that for each color class $C_i$, $N[C_i]$ contains representatives of every color class. The maximum number of color classes in a proper achromatic partition of $G$ is the achromatic number of $G$, and is denoted $\psi(G)$.

Let $\pi = \{V_1, V_2, \ldots, V_k\}$ be a partition of the vertices $V$ of a graph $G$ into distinct color classes $V_i$. For ease of discussion, if the vertices of a set $S$ are assigned colors, then we say that $S$ contains these assigned colors. Let $\text{deg}_\pi[v] = |\{i : N[v] \cap V_i \neq \emptyset\}|$; that is, $\text{deg}_\pi[v]$ equals the number of different colors assigned to vertices in the closed neighborhood of $v$ by the partition $\pi$. A (neighborhood-restricted) $[\leq k]$-coloring of $G$ is a $\pi$ partition of the vertices of $G$ wherein $\text{deg}_\pi[v] \leq k$ for all $v \in V$ [5]; that is, every closed neighborhood contains at most $k$ different colors. Figure 1 is an example of a $[\leq k]$-coloring. The $[\leq k]$-achromatic number $\psi_{[\leq k]}(G)$ is the maximum order of a $[\leq k]$-coloring of $G$; that is, $\psi_{[\leq k]}(G)$ is the maximum number of colors in any $[\leq k]$-coloring of $G$. If $\pi$ is a $[\leq k]$-coloring of $G$ with $\psi_{[\leq k]}(G)$ colors, then we say that
\( \pi \) is a \( \psi_{\leq k}(G) \)-coloring. Note that the trivial partition \( \pi = \{V\} \) is a \( [\leq k] \)-coloring for every integer \( k \geq 1 \), so \( \psi_{\leq k}(G) \geq 1 \) is defined for all graphs \( G \) and all positive integers \( k \).

**Figure 1:** Achromatic coloring of the graph \( P_6 \)

The main focus in this thesis is to consider the special case of \( [\leq k] \)-colorings where \( k = 2 \). We develop a Nordhaus-Gaddum type result for \( \psi_{\leq 2}(G) \) and improve upon a known upper bound for \( \psi_{\leq 2}(G) \). We further characterize all extremal trees in terms of a previously established upper bound on \( \psi_{\leq 2}(G) \) in terms of \( n \).
Bujtás, Sampathkumar, Tuza, Subramanya, and Dominic [3] considered \(3\)-consecutive \(C\)-colorings, which they defined to be a mapping \(\phi : V(G) \rightarrow \mathbb{N}\) such that there exists no 3-colored path in \(G\). This restriction is equivalent to our restriction of the number of distinct colors present in the closed neighborhood of a vertex \(v\) for the special case where \(k = 2\). They gave the following upper bound on \(\psi_{\leq 2}(G)\).

**Theorem 2.1** [3] For any graph \(G = (V, E)\) of order \(n\) and minimum degree \(\delta\), we have \(\psi_{\leq 2}(G) \leq \left\lfloor \frac{2n}{\delta + 1} \right\rfloor\).

In a graph \(G = (V, E)\), a set \(S \subset V\) is a neighborhood set if \(\cup_{v \in S}(N(v)) = G\), where \((N(v))\) is the subgraph induced by \(N[v]\), the closed neighborhood of \(v\). The neighborhood number of a graph \(G\), denoted by \(n_0(G)\), is the minimum cardinality of a neighborhood set in \(G\).

**Theorem 2.2** [3] Let \(G\) be a connected graph. Then, \(\psi_{\leq 2}(G) \leq n_0(G) + 1\). Further, for a tree \(T\), \(\psi_{\leq 2}(T) = n_0(T) + 1\).

**Theorem 2.3** [3] For any connected graph \(G\), \(\psi_{\leq 2}(G) \leq 2\gamma(G)\).

**Theorem 2.4** [3] A connected graph \(G = (V, E)\) has a 3-consecutive \(C\)-coloring with exactly three colors; that is, \(\psi_{\leq 2}(G) \geq 3\) if and only if its diameter is at least 3.

And finally, Bujtás et al. in [3] showed that determining whether a graph \(G\) has \(\psi_{\leq 2}(G) = 3\) or \(\psi_{\leq 2}(G) = 4\) is solvable in polynomial time.

Bujtás, Sampathkumar, Tuza, Dominic, and Pushpalatha [2] considered the case where the star subgraph for each vertex \(v\) contains at most \(k\) colors. This restriction
is equivalent to our restriction on the number of colors present in $N[v]$ for all $v \in G$, $k \in \mathbb{N}$.

Goddard and Xu [6] expanded on the work in [3], calling the colorings forbidden rainbow colorings. A subgraph is said to be rainbow if under a given coloring, its vertices receive distinct colors. A coloring having no rainbow subgraph $F$ is called a no-rainbow-$F$ coloring [6]. In the particular case where $F$ is a $P_3$, a no-rainbow-$P_3$ coloring is equivalent to a neighborhood-restricted $[\leq 2]$-achromatic coloring. More generally, for $F = K_{1,k}$, a no-rainbow-$K_{1,k}$ coloring is equivalent to a neighborhood-restricted $[\leq k]$ achromatic coloring. Goddard and Xu [6] defined the maximum cardinality of a no-rainbow-$F$ coloring of a graph $G$ as the $F$-upper chromatic number of $G$, denoted $NR_F(G)$. Thus, $NR_{K_{1,k}}(G) = \psi[\leq k](G)$, and $NR_{P_3}(G) = \psi[\leq 2](G)$.

Goddard and Xu [6] gave the following bound on $\psi[\leq 2](G)$ in terms of the diameter of $G$ and the order of $G$.

**Theorem 2.5** [6] For any graph $G$, $\psi[\leq 2](G) \geq \frac{\text{diam}(G)}{2} + 1$, and for any non-empty graph $G$, $\psi[\leq 2](G) \geq \rho(G) + 1$.

**Theorem 2.6** [6] For a connected graph $G$ of order $n$, $\psi[\leq 2](G) \leq \lfloor n/2 \rfloor + 1$.

**Theorem 2.7** [6] For a connected graph $G$ of order $n$, then $\psi[\leq 2](\text{cor}(G)) = |n| + 1$.

To build on the previous complexity result in [3], Goddard and Xu [6] showed that computing the $P_3$-upper chromatic number of $G$ is equivalent to computing the packing number of $G$. Thus, computing $NR_{P_3}(G)$ is NP-hard.
3 MAIN RESULTS

3.1 Background and Aims

The following bounds in terms of diameter are known.

Observation 3.1 [5, 6] For any connected graph $G$ with diameter $diam(G)$,

(i) $\psi_{\leq 2}(G) \geq \lceil diam(G)/2 \rceil + 1$, and

(ii) $\psi_{\leq 3}(G) \geq diam(G) + 1$.

Theorem 3.2 [3] A nontrivial connected graph $G$ has $\psi_{\leq 2}(G) = 2$ if and only if $diam(G) \leq 2$.

In Section 2, we consider the diameter of graphs and determine some Nordhaus-Gaddum type results for $\psi_{\leq 2}(G)$. Another lower bound in terms of the 2-packing number is found in [6].

Theorem 3.3 [6] For a graph $G$, $\psi_{\leq 2}(G) \geq \rho(G) + 1$.

The graphs attaining the bound of Theorem 3.3 were characterized in [5] as follows.

Theorem 3.4 [5] For any isolate-free graph $G$, $\psi_{\leq 2}(G) \geq \rho(G) + 1$ with equality if and only if $G$ has a $\psi_{\leq 2}(G)$-coloring in which at least one color class dominates $G$.

The following upper bound on $\psi_{\leq 2}(G)$ in terms of the domination number is given in [3].

Theorem 3.5 [3] For any graph $G$, $\psi_{\leq 2}(G) \leq 2\gamma(G)$. 
It is known [7] that the 2-packing number is a lower bound on the domination number of any graph $G$, that is, $\rho(G) \leq \gamma(G)$. In this section, we will characterize the graphs attaining the bound of Theorem 3.5 and improve the bound by showing that, in fact, $\psi_{[\leq 2]}(G) \leq 2\rho(G)$. Hence, we have that $\rho(G) + 1 \leq \psi_{[\leq 2]}(G) \leq 2\rho(G)$. We show every value in this range can be achieved by trees.

An upper bound on $\psi_{[\leq 2]}(G)$ in terms of the order $n$ of a graph $G$ was determined by Goddard, et al. [6].

**Theorem 3.6** [6] For a connected graph $G$ of order $n$, $\psi_{[\leq 2]}(G) \leq \lfloor(n + 2)/2\rfloor$.

Figure 2 gives another example of a $[\leq k]$-coloring of the graph $K_4 \circ K_1$. Since $\rho(K_4 \circ K_1) = 4$ and $n = 8$, Theorem 3.6 and Theorem 3.3 give that $\psi_{[\leq 2]}(K_4 \circ K_1) = 5$. Thus, the coloring in Figure 2 is also a $\psi_{[\leq k]}(G)$-coloring.

![Figure 2: Achromatic coloring of the corona graph $K_4 \circ K_1$](image)

In Section 3, we give a constructive characterization of the extremal trees for the bound of Theorem 3.6. Finally, in Section 4, we close with some open problems.
3.2 Diameter

First we obtain a bound on the $[\leq 2]$-achromatic number of $G$ by considering the diameter of its complement $\overline{G}$. Note that the diameter of a disconnected graph $G$ is defined to be $\text{diam}(G) = \infty$.

**Proposition 3.7** If $G$ is a graph and $\text{diam}(\overline{G}) \geq 3$, then $\psi_{[\leq 2]}(G) \leq 3$.

**Proof.** Since $\text{diam}(\overline{G}) \geq 3$, there exists two vertices, say $u$ and $v$, in $\overline{G}$ that are at least distance 3 apart. In $G$, $u$ and $v$ are adjacent and $\{u, v\}$ dominates $G$. Let $\pi$ be any $\psi_{[\leq 2]}(G)$-coloring. If $u$ and $v$ are colored the same color, say $c_1$, then any vertex of $N(u)$ can be colored at most one color different from $c_1$ and likewise for any vertex in $N(v)$. Hence, $\psi_{[\leq 2]}(G) \leq 3$. If $u$ and $v$ are colored different colors, say $c_1$ and $c_2$, then every vertex of $N(u) \cup N(v)$ must be colored $c_1$ or $c_2$ as well. Thus, $\psi_{[\leq 2]}(G) < 3$. □

Theorem 3.2 and Proposition 3.7 imply the following.

**Corollary 3.8** If $G$ is a nontrivial graph, then $\psi_{[\leq 2]}(G) = 2$ or $\psi_{[\leq 2]}(\overline{G}) \leq 3$.

Our next result establishes a limit on the number of color classes in any $\psi_{[\leq 2]}(G)$-coloring that can be dominating sets.

**Proposition 3.9** For any $\psi_{[\leq 2]}(G)$-coloring of a graph $G$, at most two color classes are dominating sets of $G$. Furthermore, if two color classes dominate a connected graph $G$, then $\psi_{[\leq 2]}(G) = 2$.

**Proof.** Clearly, if three color classes in any $\psi_{[\leq 2]}(G)$-coloring are dominating sets of $G$, every vertex in $G$ has a least three different colors in its closed neighborhood. Thus, no $\psi_{[\leq 2]}(G)$-coloring has more than two color classes that dominate.
Assume that a $\psi_{\leq 2}(G)$-coloring has two dominating color classes, say $V_1$ and $V_2$. Then each vertex in $V_i$ has a neighbor in $V_{3-i}$, implying that no vertex in $V_i$ for $i \in \{1, 2\}$ has a neighbor in $V \setminus (V_1 \cup V_2)$. Since $G$ is connected, it follows that $V \setminus (V_1 \cup V_2) = \emptyset$, and so $\{V_1, V_2\}$ is a partition of $V$. Hence, $\psi_{\leq 2}(G) = 2$. $\square$

Proposition 3.9 and Theorem 3.2 imply that for a connected graph $G$ with $\text{diam}(G) \geq 3$, a $\psi_{\leq 2}(G)$-coloring has at most one color class that dominates $G$.

Notice the operation of adding a new vertex and joining it to every vertex in an existing graph $H$ yields a new graph $G$ with $\psi_{\leq 2}(G) = 2$. Thus, for any graph $H$ with $\psi_{\leq 2}(H) \geq 3$, there exists a graph $G$ having $H$ as an induced subgraph and $\psi_{\leq 2}(G) = 2 < \psi_{\leq 2}(H)$. On the other hand, let $H$ be a graph having $\text{diam}(H) = 2$. By Theorem 3.2, $\psi_{\leq 2}(H) = 2$. Let $u$ and $v$ be vertices at distance 2 apart in $H$ and add a new vertex, say $v'$, and edge $vv'$, to form graph $G$. Then $\text{diam}(G) \geq 3$, and by Theorem 3.2, $\psi_{\leq 2}(G) \geq 3 > \psi_{\leq 2}(H)$. Hence, there is no inequality between the $[\leq 2]$-achromatic number of a graph $G$ and the $[\leq 2]$-achromatic number of an induced subgraph of $G$.

The following Nordhaus-Gaddum type results are proved for general $k$ in [2]. We state the theorem for the special case of $k = 2$.

**Theorem 3.10** [2] For a graph $G$ of order $n$ and its complement $\overline{G}$, $\psi_{\leq 2}(G) + \psi_{\leq 2}(\overline{G}) \leq n + 3$.

We note that if $G$ is non-trivial, and both $G$ and $\overline{G}$ are connected, then an improved Nordhaus-Gaddum type result follows directly from Theorem 3.6 and Corollary 3.8:
**Corollary 3.11** If \( G \) is non-trivial, and \( G \) and \( \overline{G} \) are connected graphs of order \( n \geq 2 \), then \( \psi_{\leq 2}(G) + \psi_{\leq 2}(\overline{G}) \leq \lfloor (n + 2)/2 \rfloor + 3 \).

### 3.3 2-Packing Number

First we characterize the graphs attaining the bound of Theorem 3.5.

**Theorem 3.12** A graph \( G \) has \( \psi_{\leq 2}(G) = 2\gamma(G) \) if and only if every \( \gamma(G) \)-set \( S \) is an efficient dominating set such that for every vertex \( v \in S \), the following hold:

1. if \( u \in N(v) \), then \( u \) is distance 2 from at most one vertex in \( S \setminus \{v\} \), and

2. there exists a vertex \( u \in N(v) \) such that \( d(u,x) \geq 3 \) for every \( x \in V \setminus N[v] \).

**Proof.** To characterize graphs attaining the bound of \( 2\gamma(G) \), assume that \( G \) is a graph with \( \psi_{\leq 2}(G) = 2\gamma(G) \). Let \( S = \{v_1, v_2, ..., v_\gamma\} \) be any \( \gamma(G) \)-set, and let \( \pi \) be a \( \psi_{\leq 2}(G) \)-coloring. Since \( S \) dominates \( G \) and every vertex of \( S \) can have at most two colors from \( \pi \) in its closed neighborhood, it follows that \( N[v_i] \) contains exactly two colors and these colors are not contained in \( V \setminus N[v_i] \) for \( 1 \leq i \leq \gamma(G) \). Hence, \( N[v_i] \cap N[v_j] = \emptyset \) for all \( v_i, v_j \in S \) for \( i \neq j \). In other words, \( S \) is a 2-packing, and so \( S \) is an efficient dominating set. Among the vertices in \( N(v_i) \) colored different from \( v_i \), select one, say \( u_i \). Since \( u_i \) and \( v_i \) are colored differently under \( \pi \), every neighbor of \( u_i \) must be colored one of the two colors assigned to \( u_i \) and \( v_i \), that is, \( N[u_i] \subseteq N[v_i] \). In particular, \( u_i \) has no neighbor in \( V \setminus N[v_i] \). To see that \( d(u_i,x) \geq 3 \) for all \( x \in V \setminus N[v_i] \), note that if \( d(u_i,x) = 2 \) for some vertex \( x \in V \setminus N[v_i] \), then the common neighbor of \( u_i \) and \( x \), say \( w \), is in \( N(v_i) \). But then \( N(w) \) contains three different colors under \( \pi \), a contradiction. Now suppose that some vertex, say \( y \), in
\( N(v_i) \) is adjacent to a vertex in \( N(v_j) \) and a vertex in \( N(v_k) \), where \( i, j, \) and \( k \) are distinct. Then \( y \) has at least three colors in its closed neighborhood, a contradiction. Hence, no vertex in \( N(v_i) \) is at distance 2 from two or more vertices in \( S \setminus \{v_i\} \) for \( 1 \leq i \leq \gamma(G) \).

For the sufficiency, assume that \( S = \{v_1, v_2, \ldots, v_k\} \) is an efficient dominating set of \( G \). As proved in [1], \( |S| = k = \gamma(G) \) and \( S \) is a packing. Assume that \( S \) satisfies the property of the theorem, that is, no vertex in \( N(v_i) \) is distance 2 from two or more vertices in \( S \setminus \{v_i\} \) for \( 1 \leq i \leq \gamma(G) \), and for every \( v_i \in S \), there exists some \( u_i \in N(v_i) \) such that \( d(u_i, x) \geq 3 \) for every \( x \in V \setminus N[v_i] \). For \( 1 \leq i \leq k \), select such a \( u_i \) for \( v_i \) and assign the color \( i \) to the vertices in \( N[v_i] \setminus \{u_i\} \) and the color \( i + k \) to the vertex \( u_i \). Note that for \( 1 \leq i \leq k \), \( N[v_i] \) and \( N[u_i] \) contain only the colors \( i \) and \( i + k \).

We claim that every vertex in \( N(v_i) \setminus \{u_i\} \) also has at most two colors in its closed neighborhood. To see this, assume that \( x_i \in N(v_i) \setminus \{u_i\} \). Clearly, if \( N[x_i] \subseteq N[v_i] \), then \( N[x_i] \) contains only the colors \( i \) and \( i + k \) and the claim holds. First assume that \( x_i \) is adjacent to \( u_i \). Since \( u_i \) is at distance three or more from every vertex in \( V \setminus N[v_i] \), it follows that \( x_i \) has no neighbor in \( V \setminus N[v_i] \), that is, \( N[x_i] \subseteq N[v_i] \). Next assume that \( x_i \) is not adjacent to \( u_i \). Thus, every vertex in \( N[x_i] \cap N[v_i] \) is colored \( i \). If \( x_i \) has no neighbor in \( V \setminus N[v_i] \), then the claim holds. Thus, assume \( x_i \) has a neighbor \( w_j \in N[v_j] \) for some \( j \neq i \). Since \( S \) is a packing and \( x_i \) is at distance 2 from at most one vertex in \( S \setminus \{v_i\} \), it follows that \( N[x_i] \subseteq (N[v_i] \setminus \{u_i\}) \cup N(v_j) \). Further, by our choice of \( u_j \), we deduce that \( w_j \neq u_j \). Therefore, every vertex in \( N[x_i] \) is colored either \( i \) or \( j \), so \( N[x_i] \) contains at most two colors. Hence, this coloring is a \([\leq 2]\)-coloring with order \( 2|S| = 2\gamma(G) \). \( \square \)
For an example of a graph attaining the bound, consider the following graph $G_k$ for $k \geq 2$ constructed as follows. Begin with the corona $P_k \circ K_1$ and subdivide each edge of the $P_k$ exactly twice. See Figure 3 for an example of $G_3$. Then $\gamma(G_k) = k$ and the set of support vertices forms a $\gamma(G_k)$-set. Let $v_1, v_2, \ldots, v_k$ denote the support vertices. Coloring each $v_i$ and its non-leaf neighbors color $i$ for $1 \leq i \leq k$, and assigning color $k + i$ to the leaf neighbor of $v_i$ yields an $\psi_{[\leq 2]}(G)$-coloring with $2k = 2\gamma(G)$ colors.

Recall that as mentioned in the introduction, the 2-packing number $\rho(G)$ is a lower bound on the domination number $\gamma(G)$ for any graph $G$. Next we improve the upper bound of Theorem 3.5.

**Theorem 3.13** For any graph $G$, $\psi_{[\leq 2]}(G) \leq 2\rho(G)$.

**Proof.** Let $S$ be a $\rho(G)$-set and $\pi$ be a $\psi_{[\leq 2]}(G)$-coloring. Suppose, to the contrary, that $\psi_{[\leq 2]}(G) \geq 2\rho(G) + 1$. We note that the vertices of $S$ contain at most $\rho(G)$ colors of $\pi$. Accordingly, there are at least $\rho(G) + 1$ color classes of $\pi$ that do not contain a vertex of $S$. Let $V_1, V_2, \ldots, V_k$ where $k \geq \rho(G) + 1$ denote the color classes of $\pi$ that do not contain a vertex of $S$. We form a set $A$ by selecting one vertex, say $v_i$, from each $V_i$, for $1 \leq i \leq k$, as follows: if $V_i \cap N(S) \neq \emptyset$, then let $v_i \in V_i \cap N(S)$, else let $v_i$ be an arbitrary vertex of $V_i$. Thus, $|A| = k \geq \rho(G) + 1$. 

![Figure 3: The graph $G_3$](image)
Note that since $S$ is a maximum 2-packing, every vertex $v_i \in A$ is either in $N(S)$ or has a neighbor, say $x_i$, in $N(S)$. Let $v_i \in V_i$ and $v_j \in V_j$ be two arbitrary vertices of $A$. To show that $A$ is a packing, we show that $d(v_i, v_j) \geq 3$. Let $c_i$ denote the color of vertex $v_i$ for all $v_i \in A$, and let $c(u)$ denote the color of vertex $u$, for all $u \not\in A$.

Since $c_i \neq c_j$ and $\pi$ is a $\psi_{\leq 2}(G)$-coloring, it follows that any common neighbor $x$ of $v_i$ and $v_j$ must be colored either $c_i$ or $c_j$; else $N[x]$ would contain at least three colors. We consider three cases:

**Case 1.** $\{v_i, v_j\} \subseteq N(S)$. Let $u \in N(v_i) \cap S$ and $w \in N(v_j) \cap S$. Since no vertex of $V_i$ is in $S$, we have that $c(u) \neq c_i$. Thus, every vertex in $N(v_i)$ must be colored either $c(u)$ or $c_i$. Similarly, every vertex in $N(v_j)$ is colored either $c_j$ or $c(w)$. Since $c_j \not\in \{c_i, c(u)\}$ and $c_i \not\in \{c_j, c(w)\}$, it follows that $v_i$ and $v_j$ are not adjacent. Further, if $x$ is a common neighbor of $v_i$ and $v_j$, then $c(x) \in \{c_i, c_j\}$. But $c_i \not\in \{c_j, c(w)\}$ and $c_j \not\in \{c_i, c(u)\}$, contradicting that $x$ is a common neighbor of $v_i$ and $v_j$. See Figure 4.

![Figure 4: Theorem 3.13, Case 1](image)

**Case 2.** Without loss of generality, $v_i \in N(S)$ and $v_j \in V \setminus N[S]$. Note that since $v_j \in V \setminus N[S]$, by the manner in which we constructed set $A$, $V_j \cap N[S] = \emptyset$, so no vertex of $N[S]$ is colored $c_j$. Since $v_i \in N(S)$, there exists some vertex $u \in S$ that

...
is adjacent to \( v_i \) and \( c(u) \not\in \{c_i, c_j\} \). Further, every vertex in \( N[v_i] \) is assigned either color \( c_i \) or \( c(u) \) under \( \pi \). Since \( c_j \not\in \{c_i, c(u)\} \), \( v_i \) and \( v_j \) are not adjacent. Moreover, \( v_j \) has neighbor \( x_j \) in \( N(S) \) and \( c_j \neq c(x_j) \), implying that every vertex in \( N[v_j] \) is colored either \( c_j \) or \( c(x_j) \). Also note that \( c(x_j) \neq c_i \), else the neighbor of \( x_j \) in \( S \) must be colored either \( c_i \) or \( c_j \), a contradiction. Now \( c_i \not\in \{c_j, c(x_j)\} \) and \( c_j \not\in \{c_i, c(u)\} \), implying that \( v_i \) and \( v_j \) have no common neighbor, \( z \). See Figure 5.

\[
\begin{align*}
S: & \quad u \quad \cdots \quad c(u) \\
N(S): & \quad v_i \quad c_i \quad z \quad x_j \quad c(x_j) \\
V \setminus N[S]: & \quad v_j \quad c_j
\end{align*}
\]

Figure 5: Theorem 3.13, Case 2

**Case 3.** Consider where \( \{v_i, v_j\} \subseteq V \setminus N[S] \). By our construction of \( A \), no vertex of \( N[S] \) can be colored \( c_i \) or \( c_j \). Again, \( v_i \) has a neighbor \( x_i \) in \( N(S) \) and \( v_j \) has a neighbor \( x_j \) in \( N(S) \). Since \( c(x_i) \neq c_i \), every vertex of \( N[v_i] \) is colored either \( c_i \) or \( c(x_i) \). Similarly, every vertex of \( N[v_j] \) is colored either \( c_j \) or \( c(x_j) \). Again, \( v_i \) and \( v_j \) are not adjacent, and since \( c_i \not\in \{c_j, c(x_j)\} \) and \( c_j \not\in \{c_i, c(x_i)\} \), they have no common neighbor, \( z \). See Figure 6.

Therefore, in all three cases, \( d(v_i, v_j) \geq 3 \). Thus, \( A \) is a 2-packing of \( G \) with cardinality \( k \geq \rho(G) + 1 \), a contradiction. Hence, we conclude that \( \psi_{\leq 2}(G) \leq 2\rho(G) \).

\( \square \)
Together, Theorems 3.4 and 3.13 yield the following corollary.

**Corollary 3.14** For any graph $G$, $\rho(G) + 1 \leq \psi_{[\leq 2]}(G) \leq 2\rho(G)$.

We next show that trees exist with $[\leq 2]$-achromatic number for every value in the range established by the bounds of Corollary 3.14.

**Theorem 3.15** Let $a$ and $b$ be positive integers such that $1 \leq a \leq b$. There exists a tree $T$ such that $\rho(T) = b$ and $\psi_{[\leq 2]}(T) = a + b$.

**Proof.** Let $a$ and $b$ be positive integers such that $1 \leq a \leq b$. Let $T$ be the tree obtained from a $P_{3a} = v_1, v_2, ..., v_{3a}$ by adding a leaf vertex $b_i$ to each $v_i$ where $i \equiv 2 (mod 3)$ and attaching $b-a$ copies of $P_2$ attached to $v_{3a}$. See Figure 7 for an example where $a = 2$ and $b = 5$. It is straightforward to see that $\rho(T) = b$. Let $\pi$ be an $\psi_{[\leq 2]}(T)$-coloring. Let $B$ be the set of leaves labeled $b_i$. Note that $N[v_i]$ can contain at most two colors of $\pi$ for each $i$ where $i \equiv 2 (mod 3)$. Thus, at most $2a$ colors can be used on the vertices in $\{v_1, v_2, ..., v_{3a}\} \cup B$. For the added $P_2$’s adjacent to $v_{3a}$, at most $b-a$ new colors are possible. Hence, $\psi_{[\leq 2]}(T) \leq 2a + b - a = a + b$. 

Figure 6: Theorem 3.13, Case 3
Consider the \([\leq 2]\)-coloring of \(T\) where the vertices of the \(P_{3a}\) are colored sequentially as follows 111222...aaa, the vertices of \(B\) are colored \(a + 1\) to \(2a\), and the remaining vertices in the \(N(v_{3a})\) are colored \(a\) while their adjacent leaves are colored \(b - a\) new distinct colors. See Figure 7. This coloring has \(a + a + b - a = a + b\) colors, implying that \(\psi\left[\leq 2\right](T) \geq a + b\), and so, \(\psi\left[\leq 2\right](T) = a + b\). \(\square\)

![Figure 7: The tree \(T\) where \(a = 2\) and \(b = 5\)](image)

### 3.4 Extremal Trees for Theorem 3.6

In this section, we characterize the trees attaining the upper bound of Theorem 3.6. We say that two vertex sets \(S, T \in V(G)\) are adjacent if there exists vertices \(s \in S\) and \(t \in T\) such that \(st \in E(G)\). We first give two lemmas. We say that a vertex \(v\) is a *monochromatic vertex* under a coloring \(\pi\) if every vertex in \(N[v]\) is in the same color class of \(\pi\).

**Lemma 3.16** A graph \(G\) of order \(n\) for which \(\psi_{\leq 2}(G) = \lfloor (n + 2)/2 \rfloor\) has at most one monochromatic vertex in any \(\psi_{\leq 2}(G)\)-coloring.

**Proof.** Suppose to the contrary that there exists some graph \(G\) of order \(n\) where \(\psi_{\leq 2}(G) = \lfloor (n + 2)/2 \rfloor\) and \(G\) has a \(\psi_{\leq 2}(G)\)-coloring \(\pi\) with monochromatic vertices.
Figure 8: Consequences of having two monochromatic vertices

\(v_1\) and \(v_2\). We build the graph \(G'\) from \(G\) by adding vertices \(v'_1\) and \(v'_2\) and edges \(v_1v'_1\) and \(v_2v'_2\). Then the coloring \(\pi\) for the vertices of \(G\) along with a new color each for \(v'_1\) and \(v'_2\) yields a \([\leq 2]\)-coloring of \(G'\) with \(\psi[\leq 2](G) + 2 = \lceil (n + 2)/2 \rceil + 2\) colors. See Figure 8. Thus, \(G'\) has order \(n + 2\) and \(\psi[\leq 2](G') \geq \lceil (n+2)/2 \rceil + 2 > \lceil ((n+2)+2)/2 \rceil\), contradicting Theorem 3.6. \(\square\)

**Lemma 3.17** A tree \(T\) of order \(n\) with \(\psi[\leq 2](T) = \lceil (n + 2)/2 \rceil\) has at most one strong support vertex and that vertex supports exactly two leaves.

**Proof.** Assume to the contrary that there exists some tree \(T\) of order \(n\) for which \(\psi[\leq 2](T) = \lceil (n + 2)/2 \rceil\), and \(T\) has either two strong support vertices or some support vertex adjacent to at least 3 leaves. Let \(\pi\) be a \(\psi[\leq 2](T)\)-coloring.

**Case 1.** \(T\) has two or more strong support vertices, say \(v_1\) and \(v_2\). Let \(v_{i,1}\) and \(v_{i,2}\) be two leaf vertices adjacent to \(v_i\) for \(i \in \{1, 2\}\). By Lemma 3.16, we have that \(T\) has at most one monochromatic vertex under \(\pi\). If a support vertex is monochromatic, then the adjacent leaves are also monochromatic, so neither \(v_1\) nor \(v_2\) is monochromatic. Moreover, at most one of their adjacent leaves is monochromatic. Hence, we may assume, without loss of generality, that each of \(v_{1,2}, v_{2,1}\), and \(v_{2,2}\) has at least two colors in their neighborhoods. This implies that \(v_2\) is a different color from each of
v_{2,1} and v_{2,2}. Thus, v_{2,1} and v_{2,2} are in the same color class. Also, v_1 and v_{1,2} are in different color classes in π, and v_{1,1} is in the same color class as either v_1 or v_{1,2}.

We now build T' from T by removing the two leaves, v_{1,1} and v_{2,1}. See Figure 9. Let π' be the restriction of π on T'. Note that π' is an \([\leq 2]\)-coloring of T'. Since v_{1,1} is in the same color class as either v_1 or v_{1,2}, that color class is still represented in π'. Similarly, v_{2,1} and v_{2,2} are in the same color class in π, so that color class is also present in π'. Thus, \(|π'| = |π| = ψ[≤2](T). Hence, ψ[≤2](T') ≥ |π'| = ψ[≤2](T) = [(n + 2)/2].

However, by Theorem 3.6, we have ψ[≤2](T') ≤ [[(n + 2) - 2]/2] = [n/2] < [(n + 2)/2] = ψ[≤2](T), which is a contradiction. Thus, T does not have two or more strong support vertices.

**Case 2.** Let T have a unique strong support vertex v with at least three leaf neighbors, say v_1, v_2, and v_3. By Lemma 3.16, at most one of v_1, v_2, and v_3 is monochromatic. Without loss of generality, assume that at least v_2 and v_3 are not monochromatic. Hence, under π, v is in a different color class than v_2 and v_3, implying that v_2 and v_3 are in the same color class. Moreover, v_1 is either in the same color class as v or the same color class as v_2 and v_3.
Now we will construct \( T' \) from \( T \) by removing \( v_1 \) and \( v_2 \). See Figure 9. Let \( \pi' \) be \( \pi \) restricted to \( T' \). Since \( v_1 \) is in the same color class under \( \pi \) as either \( v \) or \( v_3 \), that color class is still represented in \( \pi' \). Similarly, \( v_2 \) and \( v_3 \) are in the same color class, so that color class is also present in \( \pi' \). Thus, \( \psi_{\leq 2}(T') \geq |\pi'| = |\pi| = \psi_{\leq 2}(T) = \lfloor (n+2)/2 \rfloor \). As before, \( \psi_{\leq 2}(T') \leq [(n-2)+2]/2 < [(n+2)/2] = \psi_{\leq 2}(T) \), yielding the contradiction. Therefore, if \( T \) has a strong support vertex, then it is adjacent to exactly two leaves. \( \Box \)

**Definition.** Let \( f(T, v) \) be the function where \( v \) is a vertex of \( T \) and we add a \( P_2 \) with vertices \( v_a \) and \( v_b \) to \( T \) via edge \( vv_a \). Let \( \mathcal{F} \) be the smallest family of graphs such that: \( \mathcal{F} \) contains \( K_1 \) and \( K_2 \), and is closed under \( f \).

**Theorem 3.18** The family \( \mathcal{F} \) is precisely the family of trees for which \( \psi_{\leq 2}(T) = \lfloor (n+2)/2 \rfloor \).

**Proof.** Note that \( K_1 \) and \( K_2 \) can trivially be colored with one and two colors, respectively, and \( \psi_{\leq 2}(K_1) = 1 = [(1+2)/2] \) and \( \psi_{\leq 2}(K_2) = 2 = [(2+2)/2] \). To show that every tree in \( \mathcal{F} \) satisfies the equality, we proceed by induction. Assume \( T \) is a tree of order \( n \) in \( \mathcal{F} \) with \( \psi_{\leq 2}(T) = [(n+2)/2] \). Let \( \pi \) be a \( \psi_{\leq 2}(T) \)-coloring, and let \( v \) be an arbitrary vertex of \( T \). Form \( T' \) from \( T \) by applying \( f(T, v) \), that is, adding a \( P_2 \) with vertices \( v_a \) and \( v_b \) to \( T \) via edge \( vv_a \). Then \( T' \) is in \( \mathcal{F} \) and \( T' \) has order \( n' = n + 2 \). Let \( v_a \) be in the same color class as \( v \) under \( \pi \), and let \( v_b \) be in some new color class, say \( C_{v_b} \). This produces a \( \leq 2 \)-coloring for \( T' \) having \( \psi_{\leq 2}(T) + 1 \) colors, so \( \psi_{\leq 2}(T') \geq \psi_{\leq 2}(T) + 1 \). See Figure 10. By Theorem 3.6, \( \psi_{\leq 2}(T') \leq [(n+4)/2] = [(n+2)/2] + 1 = \psi_{\leq 2}(T) + 1 \), implying that \( \psi_{\leq 2}(T') = \frac{n+4}{2} \).
Figure 10: Tree characterization, Part 1

Thus, \( f \) clearly preserves trees having \( \psi_{\leq 2}(T) = [(n + 2)/2] \), and every tree in \( \mathcal{F} \) has \( \psi_{\leq 2}(T) = [(n + 2)/2] \).

To show that every tree that has \( \psi_{\leq 2}(T) = [(n + 2)/2] \) is in \( \mathcal{F} \), we proceed by induction on the order of \( T \). Since \( K_1 \) and \( K_2 \) are in \( \mathcal{F} \), and \( f(K_1, v) = P_3 \) (with \( \psi_{\leq 2}(P_3) = [(3 + 2)/2] = 2 \)), let \( T \) be a tree of order at least 4 with \( \psi_{\leq 2}(T) = [(n + 2)/2] \).

By Theorem 3.2, \( \psi_{\leq 2}(G) = 2 < [(n + 2)/2] \) for any star of order \( n \geq 4 \). Hence, we may assume that \( T \) is not a star, that is, \( diam(T) \geq 3 \). Assume that any smaller tree for which \( \psi_{\leq 2}(T) = [(n + 2)/2] \) is in \( \mathcal{F} \). We next identify a set \( P \) of vertices in \( T \) that can be pruned to leave a tree \( T_p \) with \( \psi_{\leq 2}(T_p) = [(n(T_p) + 2)/2] \), and show that \( f(T_p, v) = T \).

Choose a diametral path in \( T \), labeling the vertices of this path as \( v_1, v_2, \ldots, v_k \). If \( v_2 \) is a strong support vertex, then from Lemma 3.17, it is the only such vertex. In this case, relabel the diametral path with \( v_1 = v_k, v_2 = v_{k-1}, \ldots, v_{k-1} = v_2, v_k = v_1 \). We now observe that the degree of \( v_2 \) is 2, because \( v_2 \) has only \( v_1 \) as a leaf neighbor since it is not a strong support vertex and any neighbor other than \( v_3 \) would contradict our choice of a diametral path. Since \( T \) has at most one monochromatic neighborhood,
$v_2$ is not monochromatic. Thus, either $v_1$ and $v_2$ are in the same color class, or one of $\{v_1, v_2\}$ is in the same color class as $v_3$.

Let $P = \{v_1, v_2\}$. Then $T - P$ is a tree, say $T_p$, with order $n - 2$. In removing set $P$, we have removed exactly two vertices and at most one color class from a coloring of $T$, since either $v_1$ and $v_2$ are in the same color class or $v_3$ is a representative of the color class of either $v_1$ or $v_2$. If removing set $P$ did not remove at least one color class, then $\psi_{\leq 2}(T_p) \geq \psi_{\leq 2}(T) = [(n + 2)/2]$. But $\psi_{\leq 2}(T_p) \leq \lfloor((n - 2) + 2)/2\rfloor = [n/2] < [(n + 2)/2]$. Thus, removing $P$ removed exactly one color class from $T$, so $T_p$ can be colored with $\psi_{\leq 2}(T) - 1$ colors, implying that $\psi_{\leq 2}(T_p) \geq \psi_{\leq 2}(T) - 1 = [(n+2)/2] - 1 = [n/2]$. Since $\psi_{\leq 2}(T_p) \leq \lfloor((n - 2) + 2)/2\rfloor = [n/2]$, by Theorem 3.6, $\psi_{\leq 2}(T_p) = [n/2] = \lfloor(n(T_p) + 2)/2\rfloor$. See Figure 11.

Now clearly $T \in \mathcal{F}$, since $f(T_p, v_3) = T$, with $v_a = v_2$ and $v_b = v_1$. \(\square\)
CONCLUDING REMARKS

For future study, we are interested in characterizing the connected graphs $G$ attaining $\psi_{\leq 2}(G) = \lceil \text{diam}(G)/2 \rceil + 1$, and characterizing the graphs $G$ attaining $\psi_{\leq 2}(G) = 2\rho(G)$. We are also interested in determining bounds on $\psi_{\leq k}(G)$ in terms of $\rho(G)$ for other values of $k$. And finally, we are interested in studying $[\geq k]$ chromatic colorings wherein we require at least $k$ colors to be present in each closed neighborhood.
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