Towards a Connection between Linear Embedding and the Poincaré Functional Equation.

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Towards a Connection between Linear Embedding and the Poincaré Functional Equation

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ABSTRACT

Towards a Connection between Linear Embedding and the Poincaré Functional Equation

by

Tara Michels

Several linear embeddings of the logistic equation,

\[ x_{n+1} = \alpha x_n (1 - x_n) \]

are considered, the goal being to establish a connection between linear embedding and the Poincaré Functional Equation. In particular, we consider linear embedding schemes in a classical Hardy space.
DEDICATION

I dedicate this thesis to my parents, Larry and Anita Michels, who have helped and supported me in all my educational endeavors. Also to my boyfriend Clayton Clark, who has given invaluable support and encouragement throughout the entire research and writing process. A big "thank you" to all my friends, as well, who helped me by making me take breaks to exercise and get peanut butter milkshakes.
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1 Background and Examples

In chapter one we will examine the bifurcation diagram of the logistic equation, the Poincaré Functional Equation [5] and explore an example of linear embedding in $\ell^\infty$ [8]. We examine a specific solution to the Poincaré Functional Equation of the logistic equation. Then we look at an example of a linear embedding of the logistic equation. This gives us a linear solution to a nonlinear problem.

1.1 Bifurcation Diagram and The Poincaré Functional Equation

The bifurcation diagram of the logistic equation is essential to the background of linear embedding and the Poincaré Functional Equation [8]. A bifurcation diagram is a graph of the attractors of a nonlinear discrete dynamical system that occurs for increasing values of the parameter $a$ in the logistic equation

$$x_{n+1} = ax_n(1 - x_n).$$

The parameter $a$ is plotted on the horizontal axis and the eventual behavior of an iteration is plotted on the vertical axis. The eventual behavior of an iteration depends on the value of the parameter $a$. The diagram is pictured in figure 1.

If $a < 1$ then $X_n$ approaches zero and if $1 < a < 3$ then $X_n$ approaches a fixed point. At $a = 3$ a saddle point occurs. When $a$ is between 3 and 3.44948, $X_n$ alternates between two states, a two cycle. When $a$ is about 3.44948, $X_n$ alternates between four states—a four cycle. As $a$ increases the number of cycles increases in
a process called \textit{period doubling}. This dynamical system eventually enters a chaotic state which Devaney [5] defines according to three conditions: 1. periodic points for $f$ are dense, 2. $f$ is transitive, and 3. $f$ depends sensitively on initial conditions. This period doubling continues in the system at an increasingly frequent rate until $a$ is about 3.56994 when the system enters a chaotic state. From about $a = 3.82843$ until $a = 3.841499$ we get a period three cycle. Then we get a period six cycle and again period doubling continues into chaos. The system remains chaotic until $a = 4$. When $a = 4$ the system undergoes chaos which is invariant and after $a = 4$ we get Julia sets.

Henri Poincaré realized that even when the system is chaotic, it should still be possible to find a closed form solution. To accomplish this, Poincaré [8] introduced the Functional Equation, $F(az) = f[F(z)]$ for a dynamical system, $x_{n+1} = f(x_n)$ where $F(\cdot)$ is real-valued function for which

$$F(az) = f[F(z)].$$
**Theorem 1.1** If the Poincaré Functional Equation has a solution and \( x_0 = F(z_0) \) then \( x_n = F(a^n z_0) \).

**Proof:**

For a proof by induction, let us first notice that

\[ x_0 = F(a^0 z_0). \]

Then by the induction hypothesis

\[ x_n = F(a^n z_n). \]

As a result we obtain

\[ x_{n+1} = F(a^{n+1} z_0) = F(a \cdot a^n z_0) = F(a^{n+1} z_0). \]

This shows that the solution to the Poincaré Functional Equation yields solutions to

\[ x_{n+1} = f(x_n). \]

Now if \( f(0) = 0 \) and we let \( a = f'(0) \) where \( |a| > 1 \), the Poincaré Functional Equation has a solution with \( F(0) = 0 \). If we require \( F'(0) = 1 \), then the solution is unique and entire [8].

Let’s look at an example. Let

\[ x_{n+1} = 4x_n(1 - x_n) \]

and

\[ f(x) = 4x(1 - x). \]
Since \( f(0) = 0 \) and \( f'(0) = 4 \), the equation satisfies the conditions of Theorem 1.1. That is, there exists a unique, entire function \( F(z) \) such that

\[
F(4z) = 4F(z)(1 - F(z))
\]

where \( F(0) = 0 \) and \( F'(0) = 1 \).

The Poincaré Functional Equation implies that

\[
[F(z)]^2 = F(z) - 1/4F(4z).
\]

Since \( F \) is entire, we can write it as a power series. Since \( F(0) = 0 \) and \( F'(0) = 1 \), we get a power series of \( F \) of the form,

\[
F(z) = z + c_2z^2 + c_3z^3 + c_4z^4 + \ldots
\]

Substituting into

\[
\frac{1}{4}F(4z) = F(z) - [F(z)]^2
\]

implies that

\[
\frac{1}{4}F(4z) = \frac{1}{4}(4z) + \frac{c_2(4z)^2}{4} + \frac{c_3(4z)^3}{4} + \ldots
\]

\[
= z + c_24z^2 + c_34^2z^3 + c_44^3z^4 + \ldots
\]

It immediately follows that

\[
F(z) - \frac{1}{4}F(4z) = c_2(1-4)z^2 + c_3(1-4^2)z^3 + c_4(1-4^3)z^4 + \ldots
\]

The square of the series yields,

\[
[F(z)]^2 = 1 \cdot z^2 + 2c_2z^3 + (c_2^2 + 2c_3)z^4 + (2c_4 + 2c_2c_3)z^5 + \ldots
\]
Solving for \(c_2, c_3\) and \(c_4\) yields

\[
c_2(1 - 4) = 1 = -(1/3) = \frac{-2^3}{(2 \cdot 3 \cdot 4)} = \frac{-2^3}{4!}
\]

\[
c_3(1 - 4^2) = 2c_2 = \frac{2}{(3 \cdot 3 \cdot 5)} = \frac{2^5}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} = \frac{2^5}{6!}
\]

\[
c_4(1 - 4^3) = c_2^2 + 2c_3 = \frac{-9}{(3 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \cdot 11)} = \frac{-2^7}{(2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9)} = \frac{-2^7}{8!}
\]

Substitution into the power series yields

\[
F(z) = \frac{2^1}{2!}z - \frac{2^3}{4!}z^2 + \frac{2^5}{6!}z^3 - \frac{2^7}{8!}z^4 + \ldots
\]

\[
= \frac{1}{2} \left( \frac{2^2(\sqrt{z})^2}{2!} - \frac{2^4(\sqrt{z})^4}{4!} + \frac{2^6(\sqrt{z})^6}{6!} - \frac{2^8(\sqrt{z})^8}{8!} + \ldots \right)
\]

\[
= \frac{1}{2} - \frac{1}{2} \left( 1 - \frac{(2\sqrt{z})^2}{2!} + \frac{(2\sqrt{z})^4}{4!} - \frac{2\sqrt{z}}{6!} + \frac{(2\sqrt{z})^6}{8!} + \ldots \right)
\]

\[
= \frac{1}{2} - \frac{1}{2} \cos(2\sqrt{z}).
\]

For a given \(z_0\), let

\[
x_0 = \frac{1}{2} - \frac{1}{2} \cos(2\sqrt{z_0}).
\]

Then the solution to \(x_{n+1} = 4x_n(1 - x_n)\) is

\[
x_n = F(2^n z_0),
\]

where \(F(z) = \frac{1}{2} - \frac{1}{2} \cos(2\sqrt{z})\).

Now that we have explored the Poincaré Functional Equation solution to the logistic equation, we turn to Linear Embedding as another method for achieving solutions to the logistic equation.
1.2 An Example of Linear Embedding in $\ell^\infty$

A useful tool for studying nonlinear dynamical systems is the method of linear embedding, which is a technique of defining a larger, linear system that produces solutions to a non-linear model. Linear embedding was first developed by T. Carleman [3] in response to a challenge by David Hilbert to find a way to represent nonlinear systems using the theory of linear operators over a Hilbert space. Carleman, following an idea of Poincaré, introduced the Carleman Linearization method in 1932. The technique was not widely used, however, until the mid 1960’s, when Bellman and Richardson, among others, reintroduced the method as a technique for approximating solutions to nonlinear differential equations [1].

Today, linear embedding is used primarily as an alternative to perturbation methods. In theoretical physics, Carelman Linearization is recognized as a special case of Hirota’s bilinear method, which is used to study soliton solutions of integrable systems [6, 7]. Linear embedding is also closely related to the turning point method [8, 9], which is an important and extensively used technique in the study of nonlinear dynamical systems.

We will first explore the embedding of the equation,

$$y_{n+1} = ay_n + by_n^2$$

into $\ell^\infty$.

Let’s begin, however, by exploring the fixed points and stability of the discrete dynamical system defined by the equation above. Notice that if $b = -a$, the second
order system is the logistic equation. The fixed points \([8]\) are solutions to

\[ f(y) = y. \]

Therefore the fixed points are solutions to

\[ y = ay + by^2. \]

The fixed points are therefore, \( y = 0 \) and \( y = \frac{1-a}{b} \) where \( b \neq 0 \). We will call these fixed points \( p \). In order to visualize the fixed points we use a web diagram (a pictorial representation of the fixed points on the graph of the equation about the line \( y=x \) \([4]\)), let \( a = 2 \) and \( b = 1 \). Then the fixed points are \( p = 0 \) and \( p = -1 \). The web diagram is pictured in figure 2 \([4]\).

![Figure 2: Web Diagram of the Fixed Points](image)

By the fixed point theorem \([2]\) it can be determined that \( p = 0 \) is a repelling fixed point and \( p = -1 \) is an attracting fixed point.

Now we wish to apply the Carleman Linearization technique \([2]\) to

\[ y_{n+1} = ay_n + by_n^2. \]
In order to do so, we let $Y_{n,k} = y_n^k$. Next we substitute $y_{n+1} = ay_n + by_n^2$ into

$$Y_{n+1,k} = y_{n+1}^k$$

which yields:

$$Y_{n+1} = (ay_n + by_n^2)^k.$$ 

Next we rewrite the equation as a summation to determine the coefficients of the matrix [4]:

$$Y_{n+1,k} = \sum_{j=0}^{k} \binom{k}{j} a^j y_n^j \cdot b^{k-j} y_n^{2(k-j)}$$

$$= \sum_{j=0}^{k} \binom{k}{j} a^j b^{k-j} y_n^{2(k-j)}$$

$$= \sum_{j=0}^{k} \binom{k}{j} a^j b^{k-j} \cdot Y_{n,2k-j}.$$ 

Thus, we have achieved a linear embedding. In particular, let us define

$$V_n = \begin{bmatrix} 1 \\ Y_n \\ Y_n^2 \\ Y_n^3 \\ \vdots \end{bmatrix}.$$ 

Then $V_n$ is a map of the form

$$V_n : \mathbb{R}^1 \longrightarrow \ell^\infty,$$

where $\ell^\infty$ is the vector space of bounded sequences.

Thus, we have achieved a linear recursion relation to a non-linear equation. This will be used to build the general coefficient matrix [4]. We will call this matrix $A$:
Since we let

\[ Y_{n,k} = y^k_n, \]

then \( V_n \) satisfies the following equation

\[ V_{n+1} = AV_n. \]

Moreover, fixed points of

\[ y_{n+1} = ay_n + by^2_n \]

are mapped to fixed points of

\[ V_{n+1} = AV_n. \]

To see this, let \( p \) be a fixed point of

\[ y_{n+1} = ay_n + by^2_n. \]

Then our embedding results in

\[ p \mapsto P = \begin{bmatrix} 1 \\ p^2 \\ p^3 \\ \vdots \end{bmatrix}. \]
and it is easy to show that $P = AP$. If $y_n = p$, then

$$
Y_{n+1,k} = (a y_n + b y_{n+1})^k
= (a p + b p^2)^k
= (p)^k.
$$

So, $P = AP$. Thus we have produced an infinite dimensional, linear dynamical system which yields solutions to nonlinear finite dimensional problem.

Now we will explore a specific example. Consider

$$
x_{n+1} = 4x_n(1 - x_n)
$$

which has a solution of,

$$
x_n = \frac{1}{2} - \frac{1}{2} \cos(2\sqrt{z}).
$$

If we define $z_0$ such that

$$
x_0 = \frac{1}{2} - \frac{1}{2} \cos(2\sqrt{z_0}).
$$

then solving for $\sqrt{z_0}$ yields

$$
\sqrt{z_0} = \frac{\arccos(1 - 2x_0)}{2}.
$$

This implies that

$$
x_n = \frac{1}{2} - \frac{1}{2} \cos \left(2^n \arccos(1 - 2x_0) \right).
$$

Let

$$
\omega_0 = \arccos \left(\frac{1 - 2x_0}{4}\right)
$$
then substituting $\omega_0$ in $x_0$ yields,

$$x_n = \frac{1}{2} - \frac{1}{2} \cos(2^n \cdot 2\omega_0).$$

Simplifying $x_n$ yields,

$$x_n = \frac{1}{2} - \frac{1}{2}(1 - 2\sin^2(2^n \omega_0))$$

$$x_n = \sin^2(2^n \omega_0).$$

This implies that

$$x_0 = \sin^2(\omega_0).$$

Solving for $\omega_0$ gives us

$$\omega_0 = \arcsin(\sqrt{x_0}).$$

Using the embedding scheme,

$$Y_{n,k} = y_n^k$$

then $V_n$ satisfies the following equation:

$$V_{n+1} = AV_n.$$ 

We get

$$V_n = \begin{bmatrix}
\sin^2(2^n \omega_0) \\
\sin^4(2^n \omega_0) \\
\sin^6(2^n \omega_0) \\
\vdots
\end{bmatrix}.$$

If we let $n = 1$ at the first row of $A$ and so on until $n = k$, then at $k^{th}$ row of $A$ we get,

$$0, 0, 0, 0, \ldots, a^{2n} - \binom{n}{1}a^{2n}, \binom{n}{2}a^{2n}, \ldots, \binom{n}{n}a^{2n}.$$
where there are \( n - 1 \) zeroes to begin with. Let

\[ k = \sin(2^{2n}\omega_0)^2. \]

So now,

\[
A \cdot V_n = \begin{bmatrix}
a^2 & -a^2 & 0 & \cdots \\
0 & a^4 & -2a^4 & \cdots \\
0 & 0 & a^6 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{bmatrix} \cdot \begin{bmatrix} k \\ k^2 \\ k^3 \\ \vdots \end{bmatrix}.
\]

So upon simplifying

\[ V_{n+1} = a^{2n}k^n(1 - k)^n \]

where \( a = 4 \). Substituting

\[ \sin(2^{2n}\omega_0)^2 \]

for \( k \) yields,

\[ V_{n+1} = a^{2n}(\sin(2^{2n}\omega_0))^2(1 - (\sin(2^{2n}\omega_0)^2))^n \]

this gives us the \( n^{th} \) row of the matrix \( V_{n+1} \). Therefore upon simplifying \( V_{n+1} \) we get

\[
V_{n+1} = \begin{bmatrix}
\sin^2(2^{n+1}\omega_0) \\
\sin^4(2^{n+1}\omega_0) \\
\sin^6(2^{n+1}\omega_0) \\
\vdots \\
\sin^{2n}(2^{n+1}\omega_0) \\
\vdots
\end{bmatrix}.
\]
2 Examples of Various Linear Embedding Schemes

In this chapter we will explore some examples of embedding schemes in Hardy space, \( H^2 \) [10], \( \ell_2 \) [10], and \( H^\infty \) [10]. Here the importance of embedding space chosen is shown to be vital to the solution of the problem.

2.1 Embedding in \( H^2 \)

Let us consider the Hardy space, \( H^2 \) embedding environment where

\[
H^2 = \left\{ \sum_{n=0}^{\infty} c_n z^n : \sum_{n=0}^{\infty} |c_n|^2 < \infty \right\}.
\]

The Poincaré Functional Equation is not only entire, but it can also be shown that

\[
\sum_{n=0}^{\infty} |c_n|^2 < \infty
\]

using a variation of the Riemann mapping theorem [10]. We will not revisit the proof here, but we will demonstrate it for an example. Using

\[
F(z) = \sin^2(2^n \omega_0) = \frac{1}{2} - \frac{1}{2} \cos(2\sqrt{z})
\]

we expand \( F(z) \) into a power series;

\[
\sum_{n=0}^{\infty} c_n z^n
\]

and show that it satisfies,

\[
\sum_{n=0}^{\infty} |c_n|^2 < \infty.
\]

\( F(z) \) as a power series is,

\[
F(z) = \frac{2^1}{2!} z - \frac{2^3}{4!} z^2 + \frac{2^5}{6!} z^3 - \frac{2^7}{8!} z^4 + \ldots
\]
Now
\[ \sum_{n=0}^{\infty} c_n = \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n+1}}{(2n+2)!}. \]

We need to show that
\[ \sum_{n=0}^{\infty} |c_n|^2 = \sum_{n=0}^{\infty} \frac{(2^{2n+1})^2}{((2n+2)!)^2} \]
converges. Simplifying we get
\[ \sum_{n=0}^{\infty} |c_n|^2 = \sum_{n=0}^{\infty} \frac{4^{2n+1}}{(2n+2)!} \cdot \frac{1}{(2n+2)!} = 4 \sum_{n=0}^{\infty} \frac{8^n}{(2n+2)!} \cdot \frac{1}{(2n+2)!}. \]

Notice that,
\[ \sum_{n=0}^{\infty} \frac{1}{(2n+2)!} \]
converges since
\[ \sum_{n=0}^{\infty} \frac{1}{(2n+2)!} < \sum_{n=0}^{\infty} \frac{1}{n!} = e. \]

Since when \( n = 0 \), we get \( \frac{8^0}{2!} = \frac{1}{2} < 1 \) we assume
\[ \frac{8^n}{(2n+2)!} < 1 \]
is true for \( n \) but we need to show that this is true for \( n + 1 \). Substituting \( n + 1 \) for \( n \) yields
\[ \frac{8^{n+1}}{(2(n + 1) + 2)!} < 1. \]

Simplifying we get
\[ \frac{8 \cdot 8^n}{(2n + 4)(2n + 3)(2n + 2)!} = \frac{8}{(2n + 4)(2n + 3)} \cdot \frac{8^n}{(2n + 2)!} < 1. \]

Thus,
\[ 4 \sum_{n=0}^{\infty} \frac{8^n}{(2n+2)!} \cdot \frac{1}{(2n+2)!} \leq 4 \sum_{n=0}^{\infty} \frac{1}{(2n+2)!} \leq \sum_{n=0}^{\infty} \frac{1}{n!} = 4e < \infty \]
and \( F(z) \) for \( a = 4 \) is in \( H^2 \).
2.2 Embedding in $\ell_2$

The $\ell_2$ embedding environment is given by

$$
\ell_2 = \left\{ \langle a_0, a_1, \ldots \rangle : \sum_{j=0}^{\infty} |a_j|^2 < \infty \right\}.
$$

An embedding in $\ell_2$ is given by

$$
Y_{n,k} = \frac{y^n_k}{n!}.
$$

Thus,

$$
V_k = \begin{bmatrix}
1 \\
y_k \\
y_k^2 \\
y_k^3 \\
\vdots
\end{bmatrix}.
$$

Next we substitute $y_{n+1} = ay_n + bn^2$ into

$$
Y_{k,n} = \frac{y^n_k}{n!}
$$

which yields

$$
Y_{n+1} = \frac{(ay_n + by_n^2)^k}{n!}.
$$

Next we rewrite the equation as a summation of general linear terms to determine the input terms of the matrix:

$$
Y_{n+1,k} = \sum_{j=0}^{k} \binom{k}{j} \frac{(ay_n + by_n^2)^k}{j!}.
$$

$$
= \sum_{j=0}^{k} \binom{k}{j} \frac{a(k-j)by_n^{k+j}}{j!}.
$$

$$
= \sum_{j=0}^{k} \binom{k}{j} \frac{k!(k+j)!}{j!(k-j)!} \cdot \frac{(k+j)!}{j!}a^{k-j}b^j Y_{n,j+k}.
$$
Thus, we have achieved a linear recursion that produces solutions to a non-linear equation. This will be used to build the general matrix. We will call this matrix, $A$:

$$
A = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\
0 & a & b & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 2a^2 & 2 \cdot 3!ab & \frac{4!}{3!} b^2 & 0 & 0 & \ldots \\
0 & 0 & 3!a^3 & 3! \cdot 12a^2b & 180ab^2 & \frac{6!}{3!} b^3 & \ldots \\
0 & 0 & 0 & 4!a^4 & \frac{(4!)!}{3!} a^3b & 360a^2b^2 & \ldots \\
0 & 0 & 0 & 0 & 5!a^5 & 5 \cdot 6!a^4b & \ldots \\
& & & & & 6!a^6 & \ldots \\
& & & & & \vdots & & \ddots
\end{bmatrix}
$$

Notice that if there exists $B > 0$ such that $\|y_n\| < B$ for all $n \in \mathbb{Z}$, then

$$
V_n = \begin{bmatrix}
y_n \\
\frac{y_n^2}{2!} \\
\frac{y_n^3}{3!} \\
\frac{y_n^4}{4!} \\
\vdots
\end{bmatrix}.
$$

However, the embedding into $\ell_2$ leads to a matrix in which rows increase in magnitude very rapidly. For this reason, the choice of embedding space is critical. Thus, we turn now to $\mathcal{H}^\infty(\mathbf{T})$.

### 2.3 Embedding in $\mathcal{H}^\infty$

Let us consider the $\mathcal{H}^\infty$ embedding environment where

$$
\mathcal{H}^\infty = \left\{ \sum_{n=0}^{\infty} c_n z^n : \sup_{n \in \mathbb{Z}^+} |c_n| < \infty \right\}.
$$

Let us consider an embedding in $\mathcal{H}^\infty$ for

$$
y_{n+1} = ay_n + by_n^2.
$$
Let

\[ V_n = \sum_{k=0}^{\infty} (y_n)^k z^k. \]

Then

\[ V_{n+1} = \sum_{k=0}^{\infty} (y_{n+1})^k z^k = \sum_{k=0}^{\infty} (ay_n + by_n^2)^k z^k. \]

If we let \( V_{n+1,k} \) denote the coefficients of \( z^k \), then

\[ V_{n+1,k} = (ay_n + by_n^2)^k \]

\[ V_{n+1,k} = \sum_{\ell=0}^{k} \binom{k}{\ell} a^\ell y_n^{k-\ell} b^{k-\ell} y_n^{2\ell} \]

and we can rewrite \( V_{n+1} \) as

\[ V_{n+1} = \sum_{k=0}^{\infty} \sum_{j=0}^{k} \binom{k}{j} a^{k-j} b^j V_{n,2j-k} z^k. \]

Thus, \( V_{n+1} \) can be rewritten as,

\[ V_{n+1} = \sum_{k=0}^{\infty} \sum_{j=0}^{k} \binom{k}{j} a^{k-j} b^j y_n^{k-j} b^j y_n^{2j} z^k \]

\[ = \sum_{k=0}^{\infty} \sum_{j=0}^{k} \binom{k}{j} a^{k-j} b^j y_n^{k+j} z^k. \]

The coefficients of the matrix \( A \) are given by

\[ A_{j,k} = \binom{k}{j} a^{k-j} b^j y_n^{k+j} z^k. \]

We want to show that:

\[ \sum_{c=0}^{\infty} \sum_{r=c}^{\infty} A_{c,r} = \sum_{k=0}^{\infty} \sum_{j=0}^{k} A_{j,k}. \]
To do so, let us write the coefficient in a square array

\[
\begin{array}{cccc}
A_{0,0} & A_{1,1} & A_{2,2} & \ldots \\
A_{0,1} & A_{1,2} & A_{2,3} & \ldots \\
A_{0,2} & A_{1,3} & A_{2,4} & \ldots \\
\vdots & \vdots & \vdots & \ddots
\end{array}
\]

Adding across rows and columns yields \( \sum_{c=0}^{\infty} \sum_{r=0}^{\infty} A_{c,r} \). However, adding along diagonals yields

\[ A_{0,0} + A_{0,1} + A_{1,1} + A_{0,2} + A_{1,2} + A_{2,2} + A_{0,3} + A_{1,3} + A_{2,3} + A_{3,3} + \ldots \]

Thus, it can be clearly seen that

\[
V_{n+1} = \sum_{c=0}^{\infty} \sum_{r=0}^{\infty} A_{c,r} = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} A_{j,k}.
\]

So,

\[
V_{n+1} = \sum_{c=0}^{\infty} \sum_{r=0}^{\infty} A_{c,r} = \sum_{c=0}^{\infty} \sum_{r=0}^{\infty} \binom{r}{c} a^{r-c} b^c y_n^{r+c} z^r.
\]

This allows us to use the geometric series. So

\[
V_n = \sum_{k=0}^{\infty} y_n^k = \frac{1}{1 - y_n z}.
\]

Let \( b = -a \). Then

\[
V_{n+1} = \sum_{c=0}^{\infty} \sum_{r=0}^{\infty} \binom{r}{c} a^{r-c} b^c y_n^{r+c}.
\]

\[
V_{n+1} = \sum_{c=0}^{\infty} (-1)^c y_n^c \sum_{r=0}^{\infty} \binom{r}{c} a^r z^r.
\]

Thus,

\[
V_{n+1} = \sum_{c=0}^{\infty} \sum_{r=0}^{\infty} A_{c,r} = \sum_{c=0}^{\infty} \sum_{r=0}^{\infty} \binom{r}{c} a^{r-c} b^c y_n^{r+c} z^r.
\]
3 Towards Generalizing The Poincaré Functional Equation

In this chapter we explore the embedding of the Poincaré Functional Equation in $\ell^\infty$ and how the results relate to the Poincaré Functional Equation solution.

3.1 Results

We are embedding the Poincaré Functional Equation

$$F(az) = aF(x) + bF(x)^2$$

in $\ell^\infty$. Now let

$$R_n(x) = F(x)^n.$$  

Then our linear embedding is given by

$$R_n(az) = [aF(z) + bF(z)^2]^n = \sum_{j=0}^{k} \binom{k}{j} a^j b^{k-j} R_{n,2k-j}(z).$$

Thus, we have achieved a linear embedding. In particular, let us define

$$V_n = \begin{bmatrix} R_0(z) \\ R_1(z) \\ R_2(z) \\ R_3(z) \\ \vdots \end{bmatrix}.$$  

Then $Y_z$ is a map of the form

$$V_n : \mathbb{R}^1 \longrightarrow \ell^\infty,$$

where $\ell^\infty$ is the vector space of bounded sequences.

Therefore, $Y(az) = AY(z)$. Thus, this leads us to a proposition.
Proposition 3.1 Solutions to $Y(\alpha z) = AY(z)$, where $Y(z) = \sum_{n=0}^{\infty} c_n z^n$ are a vector subspace of $\ell^\infty$ where $Y : \mathbb{R}^1 \rightarrow \mathbb{R}^n$ and $Y$ is a vector-valued function.

Proof:

Assume $U(z)$ is another solution to:

$$U(\alpha z) = AU(z).$$

Then let

$$G(z) = Y(z) + \lambda U(z),$$

so

$$G(\alpha z) = Y(\alpha z) + \lambda U(\alpha z),$$

$$G(\alpha z) = Y(z) + \lambda AU(z),$$

and

$$G(\alpha z) = A(Y(z) + \lambda U(z)).$$

Therefore,

$$G(\alpha z) = AG(z). \Box$$

Now $Y(\alpha z) = AY(z)$ where $Y(z)$ is entire in each component and $A$ is our coefficient matrix. This leads to the following theorem.

Theorem 3.2 If $Y : \mathbb{R} \rightarrow \mathbb{R}^n$ is entire in each component and if $Y(az) = AY(z)$ for some $a$, then

$$Y(z) = \sum_{n=0}^{\infty} c_n z^n$$

where $Ac_n = a^n c_n$ and all but finitely many of the $c_n$ are zero.
Proof:

First let us show that if

\[ Y(z) = \sum_{n=0}^{\infty} c_n z^n \]

where \( Ac_n = \alpha^n c_n \) then \( Y(\alpha z) = AY(z) \). Let \( Y(\alpha z) = \sum_{n=0}^{\infty} a^n c_n z^n \) then \( Ac_n = \alpha^n c_n \).

Thus, \( Y(\alpha z) = \sum_{n=0}^{\infty} Ac_n z^n = A \sum_{n=0}^{\infty} c_n z^n \) and \( Y(z) = \sum_{n=0}^{\infty} c_n z^n \). Therefore

\[ Y(\alpha z) = AY(z). \]

Suppose now that \( U(z) \) is entire in each component and is a solution to \( Y(az) = AY(z) \). Then \( U(z) = \sum_{n=0}^{\infty} d_n z^n \). Notice that \( U(0) = AU(0) \) and \( U(0) = d_0 \) so \( Ad_0 = a^0 d_0 \). Now we need to show that this is true in general. Now we know that \( U(az) = AU(z) \) so \( aU'(az) = AU'(z) \) and at the \( n^h \) derivative we have \( a^n U^{(n)}(az) = AU^{(n)}(z) \). Let \( z = 0 \). Then \( a^n U^{(n)}(0) = AU^{(n)}(0) \). Therefore, \( a^n n! d_n = A n! d_n \). Thus, \( Ad_n = \alpha^n d_n \) and all nonzero \( d_n \) are eigenvectors of \( A \). \( \square \)

Next we seek to implement these results in Maple, so we can determine how many terms to use for \( A \) before the difference in results becomes inconsequential.

3.2 Implementation of Results in Maple

We implemented these results in Maple. First we use the general matrix \( A \) where the user can input the desired size of \( A \) with the input parameter \( A\text{size} \). Next we find the eigenvectors of \( A \), noting that they all have a multiplicity of one. We substitute the variable \( z \) for the variable \( a \) and then rewrite them as a sum from one to \( A\text{size} \).

Next we find the series with constant from \( c_0 \) to \( A\text{size} - 1 \) and list the terms in the embedding matrix. Next since the eigenvectors are derivatives we take derivatives of
our results and set them equal to each other to solve for our unknown constants. Then the user can put in any desired value for the input parameters $a$ and $b$ where $a = -b$ is the example solution worked previously. Then we get the functional equation in series form truncated at $A_{size} - 1$. The initial condition to solve for $z$ is $x_0 = 0.5$. Then we calculate the points based on a chosen value of $z_0$ generated by Maple. Upon graphing these points one can see the behavior of the logistic equation based upon the input parameters $a$ and $b$ initially selected. For example if $a = 3.1$ and $b = -3.1$ we should see if a large enough value is chosen for $A_{size}$ the graph starts to behave like a two cycle, oscillating between two values.

Thus, we can see our results verified not only through the proofs but the Maple worksheet as well.


Maple Code
We generate the general matrix $A$. The user can choose the size of $A$ using the input parameter $\text{Asize}$.

```maple
> Asize:=10:
A:=matrix(Asize,Asize,0):
for i from 0 to Asize-1 by 1 do
    for j from 0 to i by 1 do
        if(i+j<Asize) then
            A[i+1,i+j+1]:= binomial(i,j)*a^(i-j)*b^(j):
        end if:
    end do;
end do;
print(A);
```

Next, we find the eigenvectors of the matrix $A$. Note that each eigenvector has a multiplicity of one.

```maple
> B := [eigenvectors(A)]:
Here are the first few eigenvector of $A$.

```maple
> B[1];B[2];B[3];
```

Here are the first few eigenvector of $A$.
\[
\begin{bmatrix}
0, 1, \frac{(-1 + a \cdot 7)}{b}, \\
\frac{1}{2 b^7 (a' + 8 a' + 12 a' + 27 a' + 32 a' + 37 a' + 30 a' + 33)} (3 a'^2 (-14 a' + 15 a' + 26 a' + a' + 16 a' + 15 a'^2)) \\
\frac{1}{b^3 (a' + 8 a' + 12 a' + 27 a' + 32 a' + 37 a' + 30 a' + 33)} (-18 a' - 8 a' - 3 a' - 9 a' + 12 a' - 11 a' + 21 a' + 12 a' + 18 a' - 21 a' + 29 a' + 18 a' - 9 a')) \\
\frac{5 a'}{(a' - a' - a' - a' - a' - a' + a' + a' + a' - a' - a' - a' + 3 + 1) (2 + a)} \\
\frac{2 b^4 (a' + 8 a' + 12 a' + 27 a' + 32 a' + 37 a' + 30 a' + 33)}{3 (a' - a' - a' - a' + a' + a' + a' - a' - a' - a' + 3 + 2 - 1)} 5 \\
\frac{2 b^5 (a' + 8 a' + 12 a' + 27 a' + 32 a' + 37 a' + 30 a' + 33)}{a^6 (a - 1) a' (a' - a' - a' - a' + a' + a' + a' - a' - a' - a' + 3 + 2 - 1)} \\
\frac{4 b^6 (a' + 8 a' + 12 a' + 27 a' + 32 a' + 37 a' + 30 a' + 33)}{0} \\
0 
\end{bmatrix}
\]
Now we list the eigenvectors as r_1 through r_Asize, and substitute a with z.

> for i from 1 to Asize by 1 do
> r[i] := convert(B[i][3][1], 'list')*subs(a=z, B[i][1]):
> end do:

Here are the first couple of r terms:

> r[1]; r[2];

$$\begin{bmatrix}
0, 1, \frac{a(a-1)}{b}, 0, 0, 0, 0, 0, 0
\end{bmatrix} z^2$$

$$\begin{bmatrix}
0, 1, \frac{(-1 + a^6)a}{b},
\end{bmatrix}$$
Now we rewrite the list as a sum.

\[
\begin{align*}
5a^4 & (a' - a^{15} - a^{14} - a^{13} - a^{12} + a^{11} + a^{10} + 2a^9 + a^8 + a^7 - a^6 - a^5 - a^4 + 1) (2 + a) \\
2b^4 & (a^6 + 8a^5 + 12a^4 + 27a^3 + 32a^2 + 37a + 30a + 33) \\
3 & (a^{20} - a^{18} - a^{17} - a^{16} + 2a^{13} + 2a^{12} + a^{11} - 9 - 2a^8 - 2a^7 + a^4 + a^3 + 2 - 1) a^5 \\
2b^5 & (a^7 + 8a^6 + 12a^5 + 27a^4 + 32a^3 + 37a^2 + 30a + 33) \\
6 & (a - 1) (a^6 - a^5 - a^4 + 2a^3 + 2a^2 + a - 2a - 2a + a^4 + a^3 + a - 1) a^5 \\
4b^6 & (a^7 + 8a^6 + 12a^5 + 27a^4 + 32a^3 + 37a^2 + 30a + 33) \end{align*}
\]

\[
0 \]
\[
\frac{7}{z} \]
\[
0 \]
\[
\frac{a(-1 + a^7)}{b} \]

Now we rewrite the list as a sum.

\[
> \text{V_em:=add( r[i], i=1..Asize );} \\
\text{Here are the first few terms of the sum.} \\
> \text{V_em:=add(r[i], i=1..3);} \\
\left[ \begin{array}{c}
0, 1, \frac{a(a - 1)}{b}, 0, 0, 0, 0, 0, 0 \\
0, 1, \frac{(-1 + 6) a}{b}, 1 \\
2b^2 (a^7 + 8a^6 + 12a^5 + 27a^4 + 32a^3 + 37a^2 + 30a + 33) \\
+ 5a^{15} + 5a^{14} + 15 - 28a^6 + 13a^{12} + 13a - 6a^{10} - 16a^9 + 11a^3 - 28a^7 + 11a^{14} + 14a^{11}) \\
1 \end{array} \right] \]
\[
\left[ \begin{array}{c}
(3a^2 (-14a^5 + 15a^{13} - 26a^8 + a^{16} + 15a^2) \\
2b^2 (7a^7 + 8a^6 + 12a^5 + 27a^4 + 32a^3 + 37a^2 + 30a + 33) \\
+ 5a^{15} + 5a^{14} + 15 - 28a^6 + 13a^{12} + 13a - 6a^{10} - 16a^9 + 11a^3 - 28a^7 + 11a^{14} + 14a^{11}) \\
(3 (-12a^{14} - 29a^{11} + 8a^{17} + 9a^8 + 9a^{16}) \\
b^3 (7a^7 + 8a^6 + 12a^5 + 27a^4 + 32a^3 + 37a^2 + 30a + 33) \\
- 18a^{12} - 8a^6 - a^9 + 12a - 11a + 21a^5 + 12a^4 + a^4 - 21a^{10} + 29a^6 + 18a^7 - 9a, \\
5a^4 (a - a - a - a - a + a + a + 2a + a + a + a - a + a - 1) (2 + a) \\
2b^4 (7a^7 + 8a^6 + 12a^5 + 27a^4 + 32a^3 + 37a^2 + 30a + 33) \\
3 (a^{20} - a^{18} - a^{17} - a^{16} + 2a^{13} + 2a^{12} + a - 2a^8 - 2a^7 + a^4 + a^3 + a^2 + 1) a^5 \\
2b^5 (7a^7 + 8a^6 + 12a^5 + 27a^4 + 32a^3 + 37a^2 + 30a + 33) \\
a^6 (a - 1) (a^{20} - a^{18} - a^{17} - a^{16} + 2a^{13} + 2a^{12} + a - 2a^8 - 2a^7 + a^4 + a^3 + a^2 - 1) a^5 \\
4b^6 (7a^7 + 8a^6 + 12a^5 + 27a^4 + 32a^3 + 37a^2 + 30a + 33) \end{array} \right] \\
0 \]
\[
\frac{7}{z} \]
\[
0 \]
\[
\frac{a(-1 + 7a)}{b} \]
Now we normalize the eigenvectors:

First we create the matrix of powers

```
> c[1]:=1:
> expr:=add(c[j]*z^j,j=1..Asize-1);
```

```plaintext
\[ \begin{bmatrix}
(a^{11} + 27 a^{10} + 89 a^9 + 204 a^8 + 327 a^7 + 536 a^6 + 660 a^5 + 811 a^4 + 756 a^3 + 705 a^2 + 495 a + 429) b^2 \\
(3 (5 a^{21} + 19 a^{20} + 61 a^{19} + 87 a^{18} + 150 a^{17} + 155 a^{16} + 148 a^{15} + 87 a^{14} + 19 a^{13} - 148 a^{12} - 237 a^{11} - 305 a^{10} - 308 a^9 - 259 a^8 - 191 a^7 - 38 a^6 + 87 a^5 + 150 a^4 + 155 a^3 + 153 a^2 + 111 a + 99) a^2 \\
(b^3 (a^{11} + 27 a^{10} + 89 a^9 + 204 a^8 + 327 a^7 + 536 a^6 + 660 a^5 + 811 a^4 + 756 a^3 + 705 a^2 + 495 a + 429) a^3 (-165 - 558 a^{14} + a^{24} - 48 a^{11} - 117 a^{17} + 558 a^8 - 290 a^{16} - 308 a^{12} - 213 a^2 - 174 a^3 + 170 a^{19} - 548 a^9 - 89 a^8 + 89 a^7 + 476 a^6 - 476 a^5 + 137 a^4 + 174 a^3 + 290 a^2 + 213 a + 27 a^3 + 335 a^6 + 549 a^7 - 171 a)) \\
(b^4 (a^{11} + 27 a^{10} + 89 a^9 + 204 a^8 + 327 a^7 + 536 a^6 + 660 a^5 + 811 a^4 + 756 a^3 + 705 a^2 + 495 a + 429) (5 a^4 (a^{22} - a^8 - a^7 - a^6 - a^5 + a^4 + a^3 + a^2 + a^1 + a^0 + a^9 - 7 - 6 - 5 - 4 + 1) (2 a^{3} + 13 a + 12 a^2 + 15),) \\
(b^5 (a^{11} + 27 a^{10} + 89 a^9 + 204 a^8 + 327 a^7 + 536 a^6 + 660 a^5 + 811 a^4 + 756 a^3 + 705 a^2 + 495 a + 429) (3 a^5 (a^{25} - a^8 - a^7 - a^6 - a^5 + a^4 + a^3 + a^2 + a^1 + a^0 + a^9 - 8 - 6 + 3 + 4 + a^3 - 1) (9 + 5 a),) \\
(b^6 (a^{11} + 27 a^{10} + 89 a^9 + 204 a^8 + 327 a^7 + 536 a^6 + 660 a^5 + 811 a^4 + 756 a^3 + 705 a^2 + 495 a + 429) (7 a^6 (a^{17} - a^8 - a^7 - a^6 - a^5 + a^4 + a^3 + a^2 + a^1 + a^0 + 2 a^9 + 2 a^8 + 2 a^7 + 2 a^6 + 2 a^5 + 2 a^4 + 2 a^3 + 2 a^2 + 2 a^1 + 2 a^0) + a^7 - a^6 - a^5 - a^4 + 1) a^6,)
```

Now we normalize the eigenvectors:

First we create the matrix of powers

```
> c[1]:=1:
> expr:=add(c[j]*z^j,j=1..Asize-1);
```
V:=[vector(Asize,0):
for i from 0 to Asize-1 by 1 do
V[i+1]:=expr^i;
end do:
V:=convert(V,'list'):
exp:=z + c_2 z^2 + c_3 z^3 + c_4 z^4 + c_5 z^5 + c_6 z^6 + c_7 z^7 + c_8 z^8 + c_9 z^9

Next, we calculate derivatives of V and of V_em. Then we solve for Lambda.

> eqset:={}:
for i from 1 to Asize-1 do:
v_ac:=convert(subs(z=0,diff(V,z[i])),'list'):
v_em:=expand(lambda[i]*convert(subs(z=0,diff(V_em,z[i])),'list')):
eq[i]:=convert(expand(v_ac-v_em),'set'):
eqset:=eqset union eq[i]:
end do:
solve(eqset,{seq(lambda[i],i=1..Asize-1),seq(c[i],i=2..Asize-1)}):
assign(%):

Now we choose value for the input parameters a and b in order to examine the behavior of the function.

> a:=3.1;b:=-3.1;
a := 3.1
b := -3.1

Now we get the Functional Equation

> F:=expr; #Approximation for the functional equation solution F(z)
F := z - 0.4761904762 z^2 + 0.1106133510 z^3 - 0.01555986495 z^4 + 0.001493843912 z^5 - 0.0001053025386 z^6 + 0.000005725380820 z^7 - 2.488223299 10^{-7} z^8 + 8.880738344 10^{-9} z^9

Solve for z0.

> x0:=0.5;
x0 := 0.5
z0:=evalf(solve(F=x0,z));

Choose a z0 and solve the functional equation for that z0.

> z0:=-3.725704613-15.86239289*I;
for i from 1 to 20 do
x[i]:=subs(z=a^i*z0,F);
end do:
\begin{verbatim}
seq(x[i], i=1..20);

z0 := -3.725704613 - 15.86239289 I
   -9.148280491 10^6 + 1.731511247 10^7 I,
   -1.138222756 10^16 + 7.169720534 10^15 I,
   -8.112396026 10^24 + 4.551711784 10^24 I,
   -5.680671242 10^33 + 3.148265791 10^33 I,
   -3.971784926 10^42 + 2.198360054 10^42 I,
   -2.776540811 10^51 + 1.536593583 10^51 I,
   -1.940954554 10^60 + 1.074148361 10^60 I,
   -1.356831573 10^69 + 7.508863577 10^68 I,
   -9.484980393 10^77 + 5.249097570 10^77 I,
   -6.630509905 10^86 + 3.669400615 10^86 I,
   -1.753081771 10^91 + 9.701756579 10^90 I
\end{verbatim}

Now we plot the function F.

> plot(F, z=x0..Asize+1);
Notice the graph seems to be approaching a two cycle for the values of a and b we chose of 3.1 and -3.1 respectively, where A_sym is 10.
VITA

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• April 2002 Lecture: Double Domination Edge Critical Graphs Presented at the Seventh Annual North Carolina Mini-Conference on Combinatorics, Graph Theory, and Computing at Appalachian State University

• March 2003 Lecture: Maximum and Minimum Perimeter Labeled Grid Graphs Presented at Clemson University MAA Mathematics Conference

• Summer 2003 Completed project on: Plotting the Lorentz Attractor, its Poincare section and Calculating the Dimension of the Fractal Generated
• Fall 2003 Thesis: Towards a Connection between Linear Embedding and the Poincaré Functional Equation

Organizations and Honors

• Mathematical Association of America

• King College Dean’s List

• Who’s Who Among American College students