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Vertices in Total Dominating Sets.

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VERTICES IN

TOTAL DOMINATING SETS

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by

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APPROVAL

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The committee read and examined his thesis, supervised his defense of it in an oral examination, and decided to recommend that his study be submitted to the Graduate Council, in partial fulfillment of the requirements for the degree of Master of Science in Mathematics.

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ABSTRACT

VERTICES IN TOTAL DOMINATING SETS

by

Robert E. Dautermann III

Fricke, Haynes, Hedetniemi, Hedetniemi, and Laskar introduced the following concept [4]. For a graph $G = (V, E)$, let ρ denote a property of interest concerning sets of vertices. A vertex u is *ρ-good* if u is contained in a {minimum,maximum} *ρ-set* in G and *ρ-bad* if u is not contained in a *ρ-set*. Let g denote the number of ρ-good vertices and b denote the number of ρ-bad vertices. A graph G is called *ρ-excellent* if every vertex in V is ρ-good, *ρ-commendable* if $g > b > 0$, *ρ-fair* if $g = b$, and *ρ-poor* if $g < b$. In this thesis the property of interest is total domination. The total domination number, γ_t, is the cardinality of a smallest total dominating set in a graph. We investigate γ_t-excellent, γ_t-commendable, γ_t-fair, and γ_t-poor graphs.

DEDICATION

To Gramma and Grampa Dautermann, whose collective dedication and perseverance permeated throughout the entire family.

ACKNOWLEDGMENTS

I would like to thank the students and faculty of East Tennessee State University. Especially those who doubted me and those who encouraged me in this endeavor (sometimes one in the same). Of course, I need to thank the number 4. It has served me well for quite a long period of time; we would all be lost without it.

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CHAPTER 1

INTRODUCTION

A *graph* $G = (V, E)$ is a finite nonempty set $V(G)$ of objects called *vertices* together with a (possibly empty) set $E(G)$ of unordered pairs of distinct vertices of G called *edges*. We will only consider simple graphs, those without directed edges or loops.

Let $|V(G)| = n$ and $|E(G)| = m$. Two vertices of a graph are *adjacent* if there exists an edge between them. The *open neighborhood* of a vertex u, denoted as $N(u)$, consists of all vertices in $V(G)$ which are adjacent to u. The *closed neighborhood* of a vertex v, is $N[v] = N(v) \cup \{v\}$. A graph in which every possible edge exists is called a *complete graph*, denoted K_n. The graph G_1 in Figure 1 is a the complete graph on four vertices, K_4. For vertices $u, v \in V(G)$, a *u-v path* is an alternating sequence of vertices and edges that begins with the vertex u and ends with the vertex v in which each edge of the sequence joins the vertex that precedes it in the sequence to the vertex that follows it in the sequence. Moreover, no vertex is repeated in this sequence. The number of edges in the sequence is considered the *length* of the path. A graph G is *connected* if for every pair of vertices in $V(G)$, there exists a path between them. A *cycle* on n vertices, denoted C_n, is a path which originates and concludes at the same vertex. The length of a cycle is the number of edges in the cycle. For example, the graph G_2 in Figure 1 is a cycle of length 4, C_4. A *tree* is a connected graph which contains no cycles. An *endvertex* is any vertex of degree 1 (that is, a vertex adjacent to exactly one other vertex). A *support vertex* is any vertex that is adjacent to at least one endvertex. A *bipartite graph* G is a graph with independent sets V_1 and V_2 where V_1 and V_2 partition $V(G)$. A *complete bipartite graph* is a bipartite graph with partite (disjoint) sets V_1 and V_2 having the added property that every vertex of V_1 is adjacent to every vertex of V_2. Complete bipartite graphs are denoted $K_{r,s}$, where $|V_1| = r$ and $|V_2| = s$.

Figure 1: Complete Graph K_4 and Cycle C_4

A set S is a *dominating set* of G if for each $v \in V(G)$, $v \in S$ or v is adjacent to a vertex in S. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. A dominating set with cardinality $\gamma(G)$ is called a γ-*set*. For a graph G with no isolated vertices, a *total dominating set* T is a set of vertices of G for which every $v \in V(G)$ is adjacent to a vertex in T. The *total domination number* $\gamma_t(G)$ is the minimum cardinality of any total dominating set of G. As before, a total dominating set with cardinality $\gamma_t(G)$ is called a γ_t-*set*. In this thesis we are concerned with vertices in total dominating sets.

For an application of total domination, consider a mathematics conference where the attendees must form a committee to schedule the presentations. It would be optimal to have a free flow of communication between the conference attendees and the committee and also among the committee members themselves. Therefore we want the committee to possess two desirable properties. First, that every non member know at least one member of the committee, for ease of communication. Second, each member of the committee should have an acquaintance on the committee, to avoid feelings of isolation and thus enhance co-operation [3]. For example, let Bill, Ted, Sara, and Marcia be four conference attendees. Suppose Bill knows only Ted and Marcia knows only Sara, but Ted and Sara know each other. Then both Ted and Sara must be on the committee, while Bill and Marcia can not be. Had Bill and Marcia been selected for the committee, then the second property would not be met and there would be a communication gap on the committee due to the isolation of both Bill and Marcia.

Consider a graph model of our conference where each person is represented by a vertex and two vertices are adjacent if the people represented by the vertices know each other. A committee with these properties is a total dominating set of the acquaintance graph of the conference attendees. If this is the smallest such committee, then we have a γ_t-set for the graph representing our conference. If we loosen the requirements and ask only for a committee comprised of individuals who collectively know every person at the conference, but not necessarily another committee member, then we have a dominating set. If this was the smallest such committee, then we would have a γ-set. In this thesis we investigate the total dominating sets of various graphs, based on the number of vertices of a graph which are contained in total dominating sets.

Fricke, Haynes, Hedetniemi, Hedetniemi, and Laskar [4] introduced the following concept. For a graph $G = (V, E)$, let ρ denote a property of interest of sets of vertices. We say that a vertex u is *ρ-good* if u is contained in a {minimum,maximum} *ρ-set* in G and *ρ-bad* if u is not contained in a *ρ-set*. Let g denote the number of ρ-good vertices and b denote the number of ρ-bad vertices. A graph G is called *ρ-excellent* if every vertex in V is ρ-good, *ρ-commendable* if $g > b > 0$, *ρ-fair* if $g = b$, and *ρ-poor* if $g < b$. The property investigated in [4] was that of dominating sets. In other words, a vertex is *γ-good* if it is contained in some γ-set and a vertex is *γ-bad* if it is contained in no γ-set. A graph G is *γ-excellent* if every vertex in $V(G)$ is γ-good, *γ-commendable* if $g > b > 0$, *γ-fair* if $g = b$, and *γ-poor* if $g < b$.

Figure 2: P_4 and $P_4 \circ K_1$ Graphs

Before introducing our problem, we illustrate this concept with some examples. Since every graph has a dominating set, obviously every vertex-transitive graph is γ-excellent. In particular, cycles and complete graphs are γ-excellent. The 1-*corona* $G \circ K_1$ associates with every vertex $v_k \in V(G)$ a vertex u_k and joins the vertices v_k and u_k with the edge $v_k u_k$. In Figure 2, a path on four vertices, P_4, and the corona of the P_4, $P_4 \circ K_1$, are represented. In this example, every vertex in $V(P_4)$ now supports an endvertex in $P_4 \circ K_1$. Consider the following two γ-sets. Let S be the set of all endvertices in $V(P_4 \circ K_1)$. Since each vertex in S is needed to dominate its support vertex, S is a γ-set. Let T be the set of all support vertices of $V(P_4 \circ K_1)$. Each vertex in T is needed to dominate its endvertex. Hence, T is a γ-set. Therefore, every vertex in $V(P_4 \circ K_1)$ is γ-good and so $P_4 \circ K_1$ is γ-excellent. In fact, using a similar argument, we can establish that all 1-coronas are γ-excellent [4].

For an example of γ-commendable graphs, consider a subdivided star $K^*_{1,t}$ with $t \geq 2$. Since every endvertex or its support must be in every γ-set, every vertex in $V(K^*_{1,t})$ is γ-good except the center. Moreover, this center vertex will never be in any γ-set and for each endvertex there exists a γ-set containing it and another γ-set containing its support vertex. Thus, $g = 2t$ and $b = 1$. Hence $g > b$, which implies that G is γ-commendable.

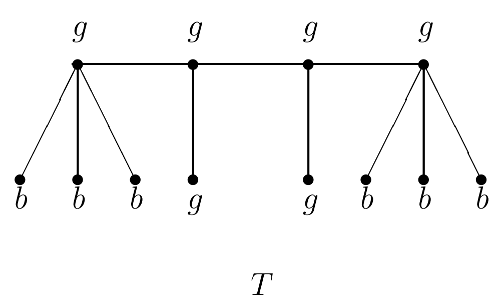

Figure 3: A γ-fair caterpillar.

For an example of a γ-fair graph, consider the caterpillar T in Figure 3. The good vertices are labeled g and the bad vertices are labeled b. For this graph, $\gamma(T) = 4$ and T is clearly γ-fair [4].

Any star $K_{1,t}$ for $t \geq 2$ is γ-poor since the only good vertex in a star is the center.

Now we return to the problem of this thesis, where the desired property is total domination. In particular, we say that a vertex is γ_t-*good* if it is in some γ_t-set and γ_t-*bad* if it is in no γ_t-set. We investigate γ_t-*excellent*, γ_t-*commendable*, γ_t-$fair$, and γ_t-*poor* graphs. First in Chapter 2 we present some known results on total domination and then give an overview of the results from [4] on γ-excellent graphs. In Chapter 3 we illustrate our concept with examples and present some new results.

CHAPTER 2

LITERATURE SURVEY AND BACKGROUND

2.1 Total Domination

Cockayne, Dawes, and Hedetniemi introduced the concept of total domination. Berge [1] presented the problem of the five queens, that is, how to place five queens on a chessboard so that every square is dominated by at least one queen [3]. It is easy to see that the solutions to this problem are dominating sets in the graph whose vertices represent the 64 squares of the chessboard and vertices a, b are adjacent if a queen may move from a to b in one move. Now extending the problem to include the property that not only must all squares be covered by a queen, but each queen must be covered by at least one other queen. This problem is that of total dominating sets, where all vertices are covered.

A total dominating set S is said to be *minimal* if when any vertex $v \in S$ is removed from S, then S is no longer a total dominating set. The following theorem gives two properties pertaining to minimal total dominating sets.

Theorem 2.1 [3] *If S is a minimal total dominating set of a connected graph $G = (V, E)$, then each $v \in S$ has at least one of the following properties:*

P_1: There exists a vertex $w \in V - S$ such that $N(w) \cap S = \{v\}$;

P_2: $< S - \{v\} >$ contains an isolated vertex.

The following theorem gives an upper bound on the total domination number of a graph. Recall that n denotes the order of a graph, or the number of vertices in the vertex set.

Theorem 2.2 [3] *If G is a connected graph with $n \geq 3$ vertices, then $\gamma_t(G) \leq 2n/3$.*

This theorem shows the best possible upper bound for $\gamma_t(G)$. Consider the path P_3. For this path, it is obvious that $\gamma_t(P_3)$=2. Since $(2 * 3)/3 = 2$, the bound is sharp.

The following proof by Henning characterizes connected graphs of order at least 3 with total domination number exactly 2/3 their order. First let us define the *k-corona*. The *k-corona* of a graph G is the graph of order $(k+1)|V(G)|$ obtained from G by identifying an endvertex v_j of a path of length k with each vertex $v \in V(G)$ and attaching this path to v by letting $v = v_j$. The resulting paths are vertex disjoint.

Theorem 2.3 [6] *Let G be a connected graph of order $n \geq 3$. Then $\gamma_t(G) = 2n/3$ if and only if G is C_3, C_6, or the 2-corona of some connected graph.*

A 4/7- *minimal graph* G is edge-minimal with respect to satisfying the following three conditions:

1: $\delta(G) \geq 2$

2: G is connected, and

3: $\gamma_t(G) \leq 4n/7$, where n is the order of G.

Henning further characterizes all 4/7-minimal graphs by the following theorem.

Theorem 2.4 [6] *If G is a connected graph of order n with minimum degree at least 2 and $G \notin \{C_3, C_5, C_6, C_{10}\}$, then $\gamma_t(G) \leq 4n/7$.*

For all paths P_n, the total domination numbers are known and easily verified. If $n = 4k$, then $\gamma_t(P_n) = 2k$ and $n \equiv 0(mod4)$. If $n = 4k+1$, then $\gamma_t(P_n) = 2k+1$ and

$n \equiv 1(mod4)$. If $n = 4k+2$, then $\gamma_t(P_n) = 2k+2$ and $n \equiv 2(mod4)$. If $n = 4k+3$, then $\gamma_t(P_n) = 2k+2$ and $n \equiv 3(mod4)$. In Chapter 3 we will use these facts to characterize all γ_t-excellent, γ_t-commendable, and γ_t-fair paths.

2.2 γ-excellent Graphs

From [4] we list several significant observations and results concerning γ-excellent graphs.

Observation 2.5 [4] *For any connected graph $G \neq K_2$, there exists a γ-set containing all the support vertices of G.*

Observation 2.6 [4] *For any γ-excellent graph G, every endvertex is in some γ-set and no endvertex is in every γ-set of G.*

For the next observation, consider a support vertex that is adjacent to two or more endvertices. In this case the support vertex must be in every γ-set. As a result, the endvertices will be in no γ-set. Hence, a graph with any support vertex adjacent to more than one endvertex is not γ-excellent.

Observation 2.7 [4] *For any γ-excellent graph G, any support vertex is adjacent to exactly one endvertex.*

This observation can be seen more clearly if one considers a star, $K_{1,t}$ for $t \geq 2$. The center vertex is adjacent to more than one endvertex. This center vertex dominates every vertex adjacent to it. It is easily shown that all stars are γ-poor.

Proposition 2.8 [4] *Every graph is an induced subgraph of a γ-excellent graph.*

Proof. Consider any graph H and let $G = H \circ K_1$, the 1-corona of a graph H. Every vertex in $V(H)$ is now a support vertex in G. Therefore, $V(H)$ is a γ-set of G. As well, the set of endvertices in G is a γ-set. Hence every vertex in $V(G)$ is in some γ-set and G is γ-excellent. Since H is an induced subgraph of G, every graph is an induced subgraph of some γ-excellent graph. □

The following proof characterizes all γ-excellent paths.

Proposition 2.9 [4] *A path P_n is γ-excellent if and only if $n = 2$ or $n \equiv 1(mod3)$.*

Proof. It is a simple exercise to see that the paths P_2 and P_n for $n \equiv 1(mod3)$ are γ-excellent. Let P_n, $n \geq 3$, be a γ-excellent path and suppose that $n \equiv 0, 2(mod3)$. If $n \equiv 0(mod3)$, then P_n has a unique γ-set, which does not include all the vertices. If $n \equiv 2(mod3)$, then no γ-set of P_n contains the third vertex on the path. □

The following theorem will show the connection between γ-excellent graphs and i-excellent, where $i(G)$ is the *independent domination number*. The independent domination number is the minimum cardinality among all independent dominating sets of a graph G, where an independent dominating set is both independent and dominating.

Theorem 2.10 [4] *If T is a γ-excellent tree, then $\gamma(T) = i(T)$ and T is i-excellent.*

CHAPTER 3

EXAMPLES AND RESULTS

This chapter contains sections for each of the following γ_t-excellent, γ_t-commendable, γ_t-fair, and γ_t-poor graphs.

We begin with the following observation involving support vertices.

Observation 3.11 *Every support vertex must be contained in every γ_t-set.*

Observation 3.12 *An endvertex adjacent to two adjacent support vertices will never be contained in any γ_t-set.*

3.1 γ_t-excellent Graphs

Every complete graph, a graph containing all possible edges, is γ_t-excellent. Since all vertices are adjacent, the selection of any two vertices will form a γ_t-set. Since complete graphs are vertex transitive, every vertex is in some γ_t-set. Moreover, all vertex transitive graphs are γ_t-excellent. This includes all cycles, C_n, and all complete bipartite graphs, $K_{r,r}$. In fact, the complete bipartite graph $K_{r,s}$ is γ_t-excellent for all r and s.

Our first proposition gives the γ_t-excellent paths. Label the vertices of the path P_n as $v_1, v_2, \ldots, v_n$.

Proposition 3.13 *Every path P_n for $n = 3$ or $n \equiv 2(mod4)$ is γ_t-excellent.*

Proof. Let P_n be a path for $n = 3$ or $n \equiv 2(mod4)$. Obviously P_3 is γ_t-excellent. For $n \equiv 2(mod4)$, $n = 4k + 2$ and $\gamma_t(P_n) = 2k + 2$. Note that P_n is γ_t-excellent for

$n = 2$ and $n = 6$. Assume some P_n is γ_t-excellent for some $n = 4k+2$. To show P_n is γ_t-excellent for $n = 4(k+1)+2$, we must verify that each of the last four vertices of $P_{4(k+1)+2}$ are in some γ_t-set. Since P_{4k+2} is assumed to be γ_t-excellent, then $v_{4k+2} \in S$ for some γ_t-set S. Since $v_{4k+2} \in S$, then v_{4k+3} is dominated. This leaves v_{4k+4}, v_{4k+5}, and v_{4k+6} to totally dominate each other. Either v_{4k+4} and v_{4k+5} or v_{4k+5} and v_{4k+6} can be used to totally dominate these three vertices. So $|S|+2 = 2k+2+2 = 2(k+1)+2$ and v_{4k+4}, v_{4k+5}, and v_{4k+6} are in some γ_t-set of $P_{4(k+1)+2}$. We need only to show that v_{4k+3} is in some γ_t-set with cardinality $2(k+1)+2$. To show this, consider a γ_t-set T for v_{4k+2} that contains v_{4k-1}. The set T exists since P_{4k+2} is γ_t-excellent. But v_{4k+2} is in T since it is a γ_t-set. So without loss of generality, let $v_{4k+2} \in S$. Then $S - \{v_{4k+1}\} \cup \{v_{4k+3}, v_{4k+4}, v_{4k+5}\}$ is a γ_t-set for v_{4k+6} and has cardinality $|S|-1+3 = |S|+2 = 2k+2+2 = 2(k+1)+2$. Hence, P_{4k+2} is γ_t-excellent. $\square$

We now consider the induced subgraphs of γ_t-excellent graphs. In particular, we show that any graph G is an induced subgraph of some γ_t-excellent graph.

As defined in Chapter 2, the generalized 2-corona is obtained from a copy of a graph G, where for each vertex $v \in V(G)$, two new vertices v' and v'', and the edges vv' and $v'v''$ are added. That is, for each vertex $v \in V(G)$, a pendant path of length 2 is added by identifying an endvertex of the new path P_3 with v. Obviously, G is an induced subgraph of each, the 1-corona $G \circ K_1$ and the 2-corona of G. Moreover, G is an induced subgraph of any k-corona of G.

Proposition 3.14 *Every graph H is an induced subgraph of a γ_t-excellent graph.*

Proof. As we have seen, every graph H is an induced subgraph of the 2-corona of

H. Let G be the 2-corona of a graph H. To see that G is γ_t-excellent, note that every γ_t-set of G must contain all the support vertices and a neighbor for each support vertex. If v is a support vertex in G, then v is in every γ_t-set of G and at least one neighbor of v is in every γ_t-set. Let S be the set of all support vertices in G and L be the set of all endvertices of G. Then $S \cup L$ and $S \cup V(H)$ are each γ_t-sets of G. Therefore G is γ_t-excellent. Since H is an induced subgraph of G, the proposition is true. □

Corollary 3.15 *There does not exist a forbidden subgraph characterization of the class of* γ_t*-excellent graphs.*

P_4 2-corona of P_4

Figure 4: P_4 and 2-corona of P_4 graphs.

The P_4 and 2-corona of the P_4 are in Figure 4. To help visualize the fore mentioned proof, one can consider these two graphs. In the 2-corona of the P_4, the four support vertices are in every γ_t-set. Let the set of support vertices be S, the set of endvertices be L, and the set of vertices in the P_4 be $V(H)$. Clearly, $S \cup L$ is a γ_t-set and $S \cup V(H)$ is as well.

The graphs G_1 and G_2 in Figure 5 are each infinite families of γ_t-excellent graphs. In each graph, the support vertices are in every γ_t-set. As well, one neighbor of each

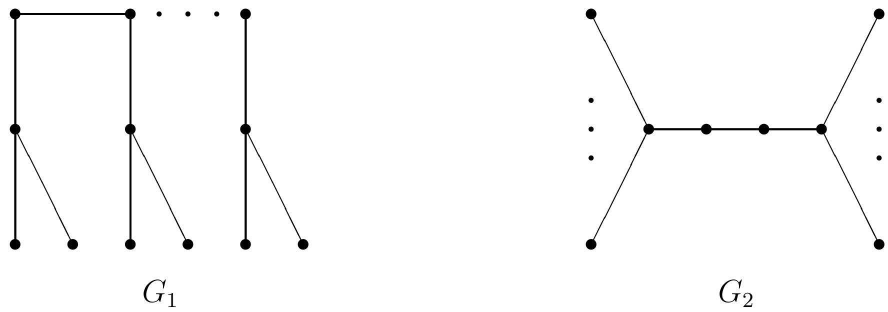

Figure 5: Infinite Families of γ_t-excellent Graphs.

support must be in every γ_t-set. It is a simple exercise to show that each of these graphs is γ_t-excellent.

3.2 γ_t-commendable Graphs

Recall that a graph is γ_t-commendable if $g > b > 0$.

Proposition 3.16 *Every 1-corona $G = H \circ K_1$ with $\Delta(H) \geq 1$ and $\delta(H) = 0$ is γ_t-commendable.*

Proof. Let $G = H \circ K_1$ with $\Delta(H) \geq 1$ and $\delta(H) = 0$. This implies that the graph H has the connected subgraph(s) $H_1, H_2, \ldots, H_k$ and at least one isolate. For each H_n, every vertex in $V(H_n)$ is now a support vertex in G. It follows that each $V(H_n)$ is in a γ_t-set of G. Further, each isolate of H is a P_2 in G. Obviously both vertices of a P_2 are in every γ_t-set. Thus, $g > n/2$ implies $g > b$. Now let u, an endvertex other than a vertex of a P_2, be in a γ_t-set S. Then the support vertex v of u must also be in S, to totally dominate u. This contradicts the minimality of S as a γ_t-set. Hence not every vertex in G is in a γ_t-set, but $g > b$. Therefore G is γ_t-commendable. □

Figure 6: $P_3 \cup K_1$ and $(P_3 \cup K_1) \circ K_1$

For example, consider $H = P_3 \cup K_1$ displayed in Figure 6. This graph is a P_3 with a disjoint singleton vertex. The graph $G = H \circ K_1$ in Figure 6 has $\gamma_t(G) = 5$. Each vertex in H is now a support vertex in G. Clearly each of these support vertices are needed to dominate their respective endvertices in G. Further, the $K_1 \circ K_1$ has total domination number 2. Since each support vertex of G is in every γ_t-set, and both vertices of any P_2 are in every γ_t-set, then G contains 5 good vertices and 3 bad vertices. Hence, G is γ_t-commendable. In general, for the 1-corona $G = H \circ K_1$, $\gamma_t(G) = |V(H)| + i$, where i is the number of isolates in H, and G has a unique γ_t-set.

The path P_5 is the path on five vertices. It is easily verifiable that $\gamma_t(P_5) = 3$. In fact, P_5 has a unique γ_t-set. Since 3 of the 5 vertices of $V(P_5)$ are in a γ_t-set, then $g > b$ and P_5 is γ_t-commendable. Notice that for P_5, $5 \equiv 1(mod 4)$. This leads us to our next proposition.

Proposition 3.17 *Every path P_n for $n \equiv 1(mod4)$ and $n \equiv 3(mod4)$, but $n \neq 3$ is γ_t-commendable.*

Proof. Let P_n be a path and $n \equiv 1(mod4)$. Recall that for $n \equiv 1(mod4)$, $n = 4k+1$ and $\gamma_t(P_n) = 2k+1$. To be γ_t-commendable, there must be more good vertices than bad, but not every vertex can be good (this would imply γ_t-excellent). Clearly there are more good vertices than bad, since $2k+1 > (4k+1)/2$. For $n = 5$, P_n has a unique γ_t-set and the endvertices are not included. Thus consider $n \geq 9$. Let the endvertex $v_1 \in S$ for some γ_t-set S. Then $v_2 \in S$ and the vertices v_1, v_2, v_3 are totally dominated. Hence, there are $n-3$ vertices remaining to totally dominate. Since $n = 4k+1$, $n-3 = 4(k-1)+2$ and $n-3 \equiv 2(mod4)$. It follows that $\gamma_t(P_{n-3}) = 2(k-1)+2 = 2k$. Therefore if an endvertex is in a γ_t-set S, then S has cardinality $2k+2$, which is a contradiction since $\gamma_t(P_{4k+1}) = 2k+1$. Hence, not every vertex is in a γ_t-set, but $g > b$. Therefore, P_n for $n \equiv 1(mod4)$ is γ_t-commendable.

Next let P_n be a path and $n \equiv 3(mod4)$ but $n \neq 3$. Clearly $g > b$ for P_n since $\gamma_t(P_n) = 2k+2 > (4k+3)/2$. Since $n \equiv 3(mod4)$, then $n = 4k+3$. Suppose $v_4 \in S$, the fourth vertex from either end of the path, for some γ_t-set S. Then there are subgraphs P_3 and $P_{4(k-1)+3}$ to totally dominate. But $\gamma_t(P_3) = 2$ and $\gamma_t(P_{4(k-1)+3}) = 2(k-1)+2 = 2k$, implying that $2k+2+1$ vertices are needed to totally dominate a P_{4k+3} if v_4 is in S. This is a contradiction since $\gamma_t(P_{4k+3}) = 2k+2 \neq 2k+3$. Hence, not every vertex is in a γ_t-set. Therefore, P_n for $n \equiv 3(mod4)$ is γ_t-commendable. $\square$

We now consider the induced subgraphs of γ_t-commendable graphs. In particular, we show that any graph H is an induced subgraph of some γ_t-commendable graph.

Proposition 3.18 *Every graph H is an induced subgraph of a γ_t-commendable graph.*

Proof. Let H be a graph. Let G be the 3-corona of H with the following property. For each vertex $v_i \in V(H)$, add vertex $v_{i'}$ and edge $v_iv_{i'}$ to the 3-corona of H. Clearly

H is an induced subgraph of G. Now we need only show that G is γ_t-commendable, or that $g > b > 0$.

Let $H_1, H_2, \cdots, H_k$ be components of H. Either $|V(H_i)| > 1$ or $|V(H_i)| = 1$. Let $|V(H_i)| = 1$. This implies that H_i is an isolate, say the vertex u_i. In G, u_i is a support vertex of a P_5. A P_5 has a unique γ_t-set and, it follows that $\gamma_t(P_5) = 3$ and every P_5 is γ_t-commendable. Therefore, for every isolated vertex in H, we have a γ_t-commendable subgraph in G.

Now let $|V(H_j)| > 1$. This implies that each $v_j \in V(H_j)$ is a support vertex in G. Moreover, each v_j is adjacent to at least 1 other support vertex, $v_h \in G$. These support vertices dominate each other, and by Observation 3.12 the endvertices $v_{j'}$ and $v_{h'}$ are not in any γ_t-set of G. Furthermore, these are the only bad vertices of G. Since each subgraph of G is γ_t-commendable, it follows that the $g > b$ for the entire graph G. Hence, G is γ_t-commendable and the proposition is true. $\square$

Corollary 3.19 *There does not exist a forbidden subgraph characterization of the class of γ_t-commendable graphs.*

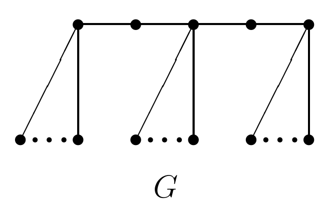

Figure 7: Infinite Family of γ_t-commendable Graphs.

The graph G in Figure 7 is another infinite family of γ_t-commendable graphs. Each support vertex in this graph is in every γ_t-set. However, the only endvertices in any γ_t-set are those which are adjacent to the outermost support vertices. The endvertices adjacent to the center support vertex are in no γ_t-set. If each support is adjacent to the same number of support vertices, the resultant graph is γ_t-commendable.

3.3 γ_t-fair Graphs

For a graph to be γ_t-fair, we must have the same number of good vertices as bad. This leads us to the following observation.

Observation 3.20 *All γ_t-fair graphs must have even order.*

Consider the path P_4. This graph has a unique γ_t-set (only the two central vertices are in a γ_t-set) and $\gamma_t(P_4) = 2$. Exactly half of the vertices in $V(P_4)$ are good and half are bad. This implies that P_4 is γ_t-fair.

We now have the following proposition concerning γ_t-fair paths.

Proposition 3.21 *A path P_n is γ_t-fair if and only if $n \equiv 0(mod4)$.*

Proof. Let P_n be a path and n $\equiv 0(mod4)$. Then $n = 4k$ and $\gamma_t(P_n) = 2k$. Since every P_{4k} has a unique γ_t-set, P_{4k} is γ_t-fair.

Assume P_n is γ_t-fair. Then from Proposition 3.13 and Proposition 3.17, $n \neq 3$ and $n \not\equiv 1, 2, 3(mod4)$. Hence, $n \equiv 0(mod4)$. □

Proposition 3.22 *Any connected graph $G = H \circ K_1$ is γ_t-fair when $|V(H)| \geq 2$.*

Proof. Let $G = H \circ K_1$ be a connected graph. Clearly $|V(G)|$=$2|V(H)|$, so the order of G is even. Since all vertices in the subgraph H of G are needed to dominate their endvertices and each vertex in H is adjacent to another vertex in H, $\gamma_t(G) = |V(H)|$, where $V(H)$ is a total dominating set of G. Notice that $V(H)$ is the set of all support vertices of G. Since all support vertices must be contained in every γ_t-set, the inclusion of any endvertex would violate the minimality of a γ_t-set. Hence, G is γ_t-fair. $\square$

Corollary 3.23 *Every connected graph H is the induced subgraph of a γ_t-fair graph.*

From Corollary 3.23 we see that there is no induced subgraph characterization of γ_t-fair graphs.

The graph $P_4 \circ K_1$ in Figure 2 is the 1-corona of the connected graph P_4. The order n of this graph is even, which is a necessary condition for γ_t-fair. Notice that the support vertices of this graph are the unique γ_t-set and no endvertex can be in any γ_t-set.

G_1 G_2

Figure 8: γ_t-fair Graphs.

Each of the graphs G_1 and G_2 in Figure 8 are γ_t-fair. For graph G_1, $\gamma_t(G_1) = 3$. The only good vertices are each of the two support vertices and the two central vertices of degree 4. The remaining 4 vertices will be in no γ_t-set. The graph G_2 is an infinite family of γ_t-fair graphs. Notice that exactly 1 of the support vertices is adjacent to exactly 2 endvertices. The center and the support vertices form the unique γ_t-set of G_2.

3.4 γ_t-poor Graphs

A graph G is γ_t-poor if $g < b$. We have shown in the previous three sections that all the paths P_n for $n \equiv 2(mod4)$ is γ_t-excellent, $n \equiv 1(mod4)$ and $n \equiv 3(mod4)$ are γ_t-commendable, and $n \equiv 0(mod4)$ are γ_t-fair. Therefore we have the following corollary.

Corollary 3.24 *There does not exist a γ_t-poor path P_n.*

We now consider the induced subgraphs of γ_t-poor graphs. In particular, we show that any graph H is an induced subgraph of some γ_t-poor graph.

Let H be a graph. Define $G = H \circ I_j$, where I_j is j isolates, as, for every vertex $v \in V(H)$, add $j \geq 2$ endvertices adjacent to v. Clearly, H is an induced subgraph of G.

Proposition 3.25 *Every graph H, where $\delta(H) \geq 1$, is an induced subgraph of a γ_t-poor graph.*

Proof. Let $G = H \circ I_j$, where $\delta(H) \geq 1$. As we have previously seen, H is an induced subgraph of G. To prove this proposition, we need only show that G is

γ_t-poor. Since $\delta(H) \geq 1$, every vertex $v \in V(H)$ has a neighbor in $V(H)$. Notice that each vertex $v \in V(H)$ is a support vertex in G. Therefore, every support vertex in G is adjacent to another support vertex in G, dominating one another and their adjacent endvertices. This implies that G has a unique γ_t-set, $V(H)$. For each vertex $v \in V(H)$, v has j support vertices adjacent to it, of which none will be in any γ_t-set. By definition, $j \geq 2$. This implies that for $|V(H)|$ good vertices, there exist $j|V(H)|$ bad vertices. Clearly there exist more bad vertices in G than good vertices. Hence, G is γ_t-poor and the proposition is true. □

Corollary 3.26 *There does not exist a forbidden induced subgraph characterization of the class of γ_t-poor graphs.*

Figure 9: Infinite Families of γ_t-poor Graphs

In each of the γ_t-poor graphs in Figure 9, all of the support vertices must be in every γ_t-set. Each of these graphs has a unique γ_t-set, the support vertices. Hence $b > g$, which implies γ_t-poor.

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